Chapter 4

A Symmetric Functional Calculus for Systems of Operators of Type ω

Brian Jefferies

ABSTRACT For a system $A = (A_1, \dots, A_n)$ of linear operators whose real linear combinations have spectra contained in a fixed sector in $\mathbb{C}$ and satisfy resolvent bounds there, functions $f(A)$ of the system A of operators can be formed for monogenic functions f having decay at zero and infinity in a corresponding sector in $\mathbb{R}^{n+1}$. In the case that the operators $A_1, \dots, A_n$ commute with each other and satisfy square function estimates in Hilbert space, the correspondence between bounded monogenic functions defined in a sector in $\mathbb{R}^{n+1}$ and bounded holomorphic functions defined in a sector in $\mathbb{C}^n$ is used to define the functional calculus $f \mapsto f(A)$ for bounded holomorphic functions f in a sector of $\mathbb{C}^n$. The treatment includes the Dirac operator on a Lipschitz surface in $\mathbb{R}^{n+1}$.

Keywords: functional calculus, plane wave formula, Dirac operator

1 Introduction

For a finite system $A = (A_1, \dots, A_n)$ of bounded linear operators acting on a Banach space X, functions $f(A)$ of the n-tuple A can be formed via the Cauchy integral formula

$$f(A) = \int_{\partial\Omega} G_x(A)\mathbf{n}(x)f(x)\, d\mu(x), \tag{1.1}$$

just under the assumption that the spectrum $\sigma(\langle A, \xi\rangle)$ of the operator $\langle A, \xi\rangle := \sum_{j=1}^n A_j\xi_j$ is a subset of $\mathbb{R}$ for every $\xi \in \mathbb{R}^n$ [6]. The Cauchy kernel $x \mapsto G_x(A)$ is defined outside a distinguished subset $\gamma(A)$ of $\mathbb{R}^n$, the open set $\Omega \subset \mathbb{R}^{n+1}$ contains $\gamma(A)$ and has a smooth oriented boundary

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$\partial\Omega$ with volume measure μ and outward unit normal $\mathbf{n}(x)$ at each point x of the boundary. The function f is defined in a neighbourhood of the closure $\bar{\Omega}$ in $\mathbb{R}^{n+1}$, takes values in the Clifford algebra $C\ell(\mathbb{C}^n)$ over $\mathbb{C}^n$ and is left monogenic in the sense of Clifford analysis. In particular, the bounded linear operator $f(A) : X \to X$ is defined for any real analytic function $f : U \to \mathbb{C}$ defined in a neighbourhood U of $\gamma(A)$ in $\mathbb{R}^n$, simply by replacing f in formula (1.1) by its left monogenic extension to an open set in $\mathbb{R}^{n+1}$ [6]. For a polynomial p in n real variables, $p(A)$ is the operator formed by substituting symmetric products in the bounded linear operators $A_1, \dots, A_n$ for the monomial expressions in p that is, we have a *symmetric* functional calculus in the n operators $A_1, \dots, A_n$. In the case $n = 1$, we obtain the usual Riesz–Dunford functional calculus for a single operator. By analogy with spectral theory in one operator, the set $\gamma(A)$ is called the *monogenic spectrum* of the n-tuple $A = (A_1, \dots, A_n)$.

The key ingredient of this approach is the Cauchy kernel $G_x(A)$. In [6], it is defined by the plane wave formula

$$G_x(A) = \frac{(n-1)!}{2}\left(\frac{i}{2\pi}\right)^n \operatorname{sgn}(x_0)^{n-1} \times \int_{S^{n-1}} (\mathbf{e}_0 + is)\left(\langle \vec{x}, s\rangle I - \langle A, s\rangle - x_0 sI\right)^{-n} ds \quad (1.2)$$

for all $x = x_0\mathbf{e}_0 + \vec{x}$ with x_0 a nonzero real number and $\vec{x} \in \mathbb{R}^n$. Here S^{n-1} is the unit $(n-1)$-sphere in $\mathbb{R}^n$, ds is surface measure and the inverse power $(\langle \vec{x}I - A, s\rangle - x_0 s)^{-n}$ is taken in the Clifford module $\mathcal{L}(X) \otimes C_{(n)}$. The spectral reality condition

$$\sigma(\langle A, \xi\rangle) \subset \mathbb{R}, \quad \text{for all} \quad \xi \in \mathbb{R}^n, \quad (1.3)$$

ensures the invertibility of $(\langle \vec{x}I - A, s\rangle - x_0 s)$ for all $x_0 \neq 0$ and $s \in S^{n-1}$ by the spectral mapping theorem. The Cauchy kernel $G_x(A)$ given by formula (1.2) coincides with the convergent series expansion of $G_x(A)$ for large $|x|$.

If we now pass to unbounded operators, then a similar analysis holds if we retain the spectral reality condition (1.3), provided that we suitably account for operator domains.

The question of forming functions of noncommuting systems of operators arises in quantum physics [14] and the connection with Clifford analysis is already apparent in [9]. Explicit calculations can be made with two hermitian matrices [7]. The application of ideas from Clifford analysis to Feynman's operational calculus is pursued in joint work of the author with G.W. Johnson [5].

The examination of functional calculi for n-tuples of operators, and formula (1.2) in particular, was also motivated by the study of the commuting n-tuple $D_\Sigma = (D_1, \dots, D_n)$ of differentiation operators on a Lipschitz surface Σ in $\mathbb{R}^{n+1}$, see [13]. In the case that Σ is just the flat surface $\mathbb{R}^n$, the operators $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$, commute with each other and are self-adjoint, otherwise, the unbounded operators D_j, $j = 1, \ldots, n$, have spectra $\sigma(D_j)$ contained in a complex sector

$$S_\omega(\mathbb{C}) = \{z \in \mathbb{C} : z \neq 0,\ |\arg(z)| \leq \omega\}$$

with an angle ω depending on the variation of the directions normal to the surface Σ.

The existence and properties of the H^∞-functional calculus for the commuting n-tuple D_Σ are now well–understood, see for example [13]. Here we show how a functional calculus for an n-tuple A of unbounded operators of type ω can be constructed directly and to apply this to the problem of forming bounded linear operators $f(D_\Sigma)$ acting on $L^p(\Sigma)$ when f is a bounded holomorphic function defined on a suitable sector in $\mathbb{C}^n$. The construction of an H^∞-functional calculus for D_Σ has applications to the solution of irregular boundary value problems. The spectral reality condition (1.3) needs to be replaced by a condition where the spectra are contained in a fixed sector of angle ω in the complex plane in order to treat systems of operators such as D_Σ. The convergence of the integral on the right–hand side of the plane wave formula (1.2) for the Cauchy kernel is then at issue.

In Section 2, it is shown how formula (1.2) for the Cauchy kernel associated with the system A of sectorial operators still works if the spectral reality condition (1.3) is replaced by a sectoriality condition with the appropriate resolvent bounds. The system D_Σ of commuting sectorial operators described above is of this type. By this means functions $f(A)$ of the operators A can be formed, provided that f is left monogenic in a sector in $\mathbb{R}^{n+1}$ and satisfies suitable decay estimates at 0 and ∞, in a fashion similar to the case of a single operator of type ω [12]. Because $G_x(A)$ is only defined for x outside a sector in $\mathbb{R}^{n+1}$, the monogenic spectrum $\gamma(A)$ is now contained in that sector in $\mathbb{R}^{n+1}$. Recall that under condition (1.3), $\gamma(A)$ is a subset of $\mathbb{R}^n \equiv \mathbb{R}^n \times \{0\}$.

A function $f(D_\Sigma)$ of the system D_Σ has a natural interpretation as a multiplier operator acting on L^p-spaces of functions defined on the Lipschitz surface Σ as well as a singular convolution operator, so the multiplier f should be a bounded holomorphic function defined on a sector in $\mathbb{C}^n$ [13], rather than a bounded monogenic function defined in a sector in $\mathbb{R}^{n+1}$.

In this work, we use the recently proven bijection [3] between bounded monogenic functions defined on a sector in $\mathbb{R}^{n+1}$ and bounded holomorphic functions defined on a sector in $\mathbb{C}^n$ to resolve this apparent dilemma about what is the appropriate function space for the symmetric functional calculus. The association between the two function spaces is via the Cauchy–Kowaleski extension to a sector in $\mathbb{R}^{n+1}$ of the restriction of the holomorphic function to $\mathbb{R}^n \setminus \{0\}$.

In Section 3 the bijection between bounded monogenic functions and bounded holomorphic functions in sectors is described. In Section 4, this enables us to form functions $f(A)$ of a commuting system A of operators acting in a Hilbert space when the single operator $\sum_{j=1}^n A_j\mathbf{e}_j$ satisfies square function estimates and f is bounded and holomorphic in a sector.

Some notation and facts from Clifford analysis [1, 2] follow.

Let $C\ell(\mathbb{C}^n)$ be the *Clifford algebra* generated over the field $\mathbb{C}$ by the standard basis vectors $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n$ of $\mathbb{R}^{n+1}$ with conjugation $u \mapsto \overline{u}$. The *generalized Cauchy–Riemann operator* is given by $D = \sum_{j=0}^n \mathbf{e}_j \frac{\partial}{\partial x_j}$.

Let $U \subset \mathbb{R}^{n+1}$ be an open set. A function $f : U \to C\ell(\mathbb{C}^n)$ is called *left monogenic* if $Df = 0$ in U and *right monogenic* if $fD = 0$ in U. The *Cauchy kernel* is given by

$$G_x(y) = \frac{1}{\sigma_n} \frac{\overline{x-y}}{|x-y|^{n+1}}, \quad x, y \in \mathbb{R}^{n+1}, x \neq y, \tag{1.4}$$

with $\sigma_n = 2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right)$ the volume of the unit n-sphere in $\mathbb{R}^{n+1}$. So, given a left monogenic function $f : U \to C\ell(\mathbb{C}^n)$ defined in an open subset U of $\mathbb{R}^{n+1}$ and an open subset Ω of U such that the closure $\overline{\Omega}$ of Ω is contained in U, and the boundary $\partial\Omega$ of Ω is a smooth oriented n-manifold, then the Cauchy integral formula

$$f(y) = \int_{\partial\Omega} G_x(y)\mathbf{n}(x)f(x)\, d\mu(x), \quad y \in \Omega$$

is valid. Here $\mathbf{n}(x)$ is the outward unit normal at $x \in \partial\Omega$ and μ is the volume measure of the oriented manifold $\partial\Omega$. An element $x = (x_0, x_1, \dots, x_n)$ of $\mathbb{R}^{n+1}$ will often be written as $x = x_0\mathbf{e}_0 + \vec{x}$ with $\vec{x} = \sum_{j=1}^n x_j\mathbf{e}_j$.

2 The plane wave decomposition

Let $A = (A_1, \dots, A_n)$ be an n-tuple of densely defined linear operators $A_j : \mathcal{D}(A_j) \to X$ acting in X such that $\cap_{j=1}^n \mathcal{D}(A_j)$ is dense in X. The space $\mathcal{L}_{(n)}(X_{(n)})$ of left module homomorphisms of $X_{(n)} = X \otimes C\ell(\mathbb{C}^n)$ is identified with $\mathcal{L}(X) \otimes C\ell(\mathbb{C}^n)$ in the natural way and becomes a right Banach module under the uniform operator topology.

If we take formula (1.2) as the definition of the Cauchy kernel $G_x(A)$, then we need to establish the convergence of the integral

$$\int_{S^{n-1}} (\mathbf{e}_0 + is)\left(\langle \vec{x}I - A, s\rangle - x_0 sI\right)^{-n} ds$$

for particular values of $x = x_0\mathbf{e}_0 + \vec{x} \in \mathbb{R}^{n+1}$. Now

$$\left(\langle \vec{x}I - A, s\rangle - x_0 sI\right)^{-1} = \left(\langle \vec{x}I - A, s\rangle + x_0 sI\right)\left(\langle \vec{x}I - A, s\rangle^2 + x_0^2 I\right)^{-1}$$

if $0 \notin \sigma\left(\langle \vec{x}I - A, s\rangle^2 + x_0^2\right)$. Thus, we need to ensure the appropriate uniform operator bounds for

$$\left(\langle \vec{x}I - A, s\rangle^2 + x_0^2 I\right)^{-1}, \quad s \in S^{n-1}$$

as $x = x_0\mathbf{e}_0 + \vec{x}$ ranges over a subset of $\mathbb{R}^{n+1}$. In the case that $\sigma(\langle A, s\rangle) \subset \mathbb{R}$ and $(\lambda I - \langle A, s\rangle)^{-1}$ is suitably bounded for all $s \in S^{n-1}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $G_{x_0\mathbf{e}_0+\vec{x}}(A)$ is defined for all $x_0 \neq 0$. First, for $0 < \nu < \frac{1}{2}\pi$, set

$$S_\nu(\mathbb{C}) = \{z \in \mathbb{C} : |\arg z| \leq \nu\} \cup \{0\}, \quad S^\circ_\nu(\mathbb{C}) = \{z \in \mathbb{C} : |\arg z| < \nu\}.$$

The set of $s \in S^{n-1}$ with nonzero coordinates s_j for every $j = 1, \ldots, n$ is denoted by S^{n-1}_0. Then S^{n-1}_0 is a dense open subset of S^{n-1} with full surface measure.

Definition 1. *Let $A = (A_1, \ldots, A_n)$ be an n-tuple of closed, densely defined linear operators $A_j : \mathcal{D}(A_j) \to X$ acting in X such that $\cap_{j=1}^n \mathcal{D}(A_j)$ is dense in X and let $0 \leq \omega < \frac{\pi}{2}$. The operator*

$$\langle A, s\rangle : \cap_{j=1}^n \mathcal{D}(A_j) \to X$$

is defined by

$$\langle A, s\rangle u = \sum_{j=1}^n s_j(A_j u) , \quad \forall s \in S^{n-1},\ u \in \cap_{j=1}^n \mathcal{D}(A_j).$$

Then A is said to be uniformly of type ω if $\sigma(\langle A, s\rangle) \subset S_\omega(\mathbb{C})$ for all $s \in S^{n-1}_0$ and for each $\nu > \omega$, there exists $C_\nu > 0$ such that

$$\|(zI - \langle A, s\rangle)^{-1}\| \leq C_\nu |z|^{-1}, \quad z \notin S_\nu(\mathbb{C}),\ s \in S^{n-1}_0. \tag{2.1}$$

If $n = 1$, the A is just said to be of type ω.

It follows that $s \mapsto \langle A, s\rangle$ is continuous on S^{n-1}_0 in the sense of strong resolvent convergence [8, Theorem VIII.1.5]. The subset S^{n-1}_0 of S^{n-1} is used here simply because $\cap_{j=1}^n \mathcal{D}(A_j)$ may be strictly contained in $\mathcal{D}(A_k)$ for $k = 1, \ldots, n$.

Now suppose that equation (2.1) is satisfied and let $z = \langle \vec{x}, s\rangle + ix_0$. Then $z \notin S^\circ_\nu(\mathbb{C})$ means that $|x_0| \geq \tan\nu |\langle \vec{x}, s\rangle|$.

First, let

$$N_\nu = \{x \in \mathbb{R}^{n+1} : x = x_0\mathbf{e}_0 + \vec{x},\ |x_0| \geq \tan\nu |\vec{x}|\ \},$$
$$S_\nu(\mathbb{R}^{n+1}) = \{x \in \mathbb{R}^{n+1} : x = x_0\mathbf{e}_0 + \vec{x},\ |x_0| \leq \tan\nu |\vec{x}|\ \},$$
$$S^\circ_\nu(\mathbb{R}^{n+1}) = \{x \in \mathbb{R}^{n+1} : x = x_0\mathbf{e}_0 + \vec{x},\ |x_0| < \tan\nu |\vec{x}|\ \}.$$

Note that if $x_0\mathbf{e}_0 + \vec{x} \in N_\nu$, then $z = \langle \vec{x}, s\rangle + ix_0 \notin S^\circ_\nu(\mathbb{C})$ for every $s \in S^{n-1}$, because $|x_0| \geq \tan\nu|\vec{x}| \geq \tan\nu|\langle \vec{x}, s\rangle|$.

The proof of the following result is straightforward and appears in [4].

Lemma 1. *Let $\omega < \nu < \frac{1}{2}\pi$. Suppose that the n-tuple A of linear operators is uniformly of type ω. Then for all $x_0\mathbf{e}_0 + \vec{x} \in N_\nu$, the integral*

$$\int_{S^{n-1}} \left\| (\langle \vec{x}I - A, s\rangle - x_0 sI)^{-n} \right\|_{\mathcal{L}_{(n)}(X_{(n)})} ds$$

converges and satisfies the bound

$$\int_{S^{n-1}} \left\| (\langle \vec{x}I - A, s\rangle - x_0 sI)^{-n} \right\|_{\mathcal{L}_{(n)}(X_{(n)})} \, ds \le \frac{C'_\nu}{|x_0|^n}.$$

Thus, if A is uniformly of type ω, then

$$x_0\mathbf{e}_0 + \vec{x} \mapsto G_{x_0\mathbf{e}_0+\vec{x}}(A) \tag{2.2}$$

is defined by the plane wave formula (1.2) for all $x_0\mathbf{e}_0 + \vec{x} \in N_\nu$ with $\omega < \nu < \frac{1}{2}\pi$. Standard arguments ensure that (2.2) is both left and right monogenic as an element of $\mathcal{L}_{(n)}(X_{(n)})$. If we denote by $\gamma(A) \subset \mathbb{R}^{n+1}$ the set of all singularities of function (2.2), then $\gamma(A) \subseteq S_\omega(\mathbb{R}^{n+1})$.

Suppose that $\omega < \nu < \frac{1}{2}\pi$, $0 < s < n$ and f is a left monogenic function defined on $S^\circ_\nu(\mathbb{R}^{n+1})$ such that for every $0 < \nu' < \theta < \nu$ there exists $C_{\theta,\nu'} > 0$ such that

$$|f(x)| \le C_{\theta,\nu'} \frac{|x|^s}{1+|x|^{2s}}, \quad x \in S^\circ_\theta(\mathbb{R}^{n+1}) \cap N_{\nu'}. \tag{2.3}$$

According to Lemma 1, for every $\omega < \nu' < \theta < \nu$, we have

$$\|G_x(A)\|.|f(x)| \le C'_{\theta,\nu'} \frac{|x|^s}{|x_0|^n(1+|x|^{2s})}, \quad x = x_0\mathbf{e}_0 + \vec{x}$$

for all $x \in S^\circ_\theta(\mathbb{R}^{n+1}) \cap N_{\nu'}$. Now if $\omega < \theta < \nu$ and

$$H_\theta = \{x \in \mathbb{R}^{n+1} : x = x_0\mathbf{e}_0 + \vec{x},\ |x_0|/|x| = \tan\theta\} \subset S^\circ_\nu(\mathbb{R}^{n+1}) \tag{2.4}$$

it follows that

$$\|G_x(A)\|.|f(x)| = O(1/|x|^{n-s}) \quad \text{as} \quad x \to 0 \quad \text{in} \quad H_\theta.$$

Hence, $x \mapsto G_x(A)\mathbf{n}(x)f(x)$ is locally integrable at zero with respect to n-dimensional surface measure μ on H_θ. Similarly,

$$\|G_x(A)\|.|f(x)| = O(1/|x|^{n+s}) \quad \text{as} \quad |x| \to \infty \quad \text{in} \quad H_\theta,$$

so $x \mapsto G_x(A)\mathbf{n}(x)f(x)$ is integrable with respect to n-dimensional surface measure on H_θ. Therefore, we define the element $f(A)$ of the module $\mathcal{L}_{(n)}(X_{(n)})$ by the formula

$$f(A) = \int_{H_\theta} G_x(A)\mathbf{n}(x)f(x)\, d\mu(x). \tag{2.5}$$

If $\psi : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ has a two-sided monogenic extension $\tilde{\psi}$ to $S^\circ_\nu(\mathbb{R}^{n+1})$ that satisfies the bound (2.3) for all $0 < \theta < \nu$, then $\tilde{\psi}(A)$ is written just as

$\psi(A)$. Formula (2.5) does just what we would expect in the noncommuting situation. For example, let p be a polynomial of degree n with

$$p(0) = 0 \quad \text{and} \quad b_\lambda(z) = p(z)(\lambda - z)^{-n-1}$$

for some $\lambda \notin S_\nu^\circ(\mathbb{C})$. Let $\xi \in \mathbb{R}^n$ and set $\phi_{\lambda,\xi}(\vec{x}) = b_\lambda(\langle \vec{x}, \xi\rangle)$ for all $\vec{x} \in \mathbb{R}^n$. Denote the two-sided monogenic extension of $\phi_{\lambda,\xi}$ to $S_\nu^\circ(\mathbb{R}^{n+1})$ by $\tilde{\phi}_{\lambda,\xi}$. Then $\tilde{\phi}_{\lambda,\xi}$ has decay at zero and infinity and we have

$$\phi_{\lambda,\xi}(A) = \tilde{\phi}_{\lambda,\xi}(A) = p(\langle A, \xi\rangle)(\lambda I - \langle A, \xi\rangle)^{-n-1}$$

is a bounded linear operator. More generally, the next result shows that formula (2.5) gives the right result when for a special class of functions f, there is a representation of the bounded linear operator $f(A)$ by an alternative formula, namely, the usual Riesz–Dunford functional calculus. Set $D_r = \{z \in \mathbb{C} : |z| < r\}$ for each $r > 0$.

Theorem 1. *Let $0 < \omega < \nu < \frac{1}{2}\pi$ and suppose that $A = (A_1, \dots, A_n)$ is an n-tuple of operators uniformly of type ω with the property that for each $\omega < \omega' < \nu$, there exist n-tuples $A^{(k)} = \left(A_1^{(k)}, \dots, A_n^{(k)}\right)$, $k = 1, 2, \dots,$ of bounded linear operators such that, with ω' replacing ω, the bound (2.1) is satisfied uniformly for $k = 1, 2, \dots$ and*

$$(zI - \langle A^{(k)}, s\rangle)^{-1} \longrightarrow (zI - \langle A, s\rangle)^{-1}, \quad z \notin S_\nu(\mathbb{C}),\ s \in S_0^{n-1}, \qquad (2.6)$$

as $k \to \infty$.

Suppose that $0 < s < 1$, $0 < \omega < \nu$, $r > 0$ and $\phi : S_\nu^\circ(\mathbb{C}) \cup D_r \to \mathbb{C}$ is a holomorphic function satisfying $\phi(0) = 0$, such that for each $0 < \nu' < \nu$, there exists $C_{\nu'} > 0$ with

$$|\phi(z)| \le \frac{C_{\nu'}}{1 + |z|^s}, \quad z \in S_{\nu'}^\circ(\mathbb{C}).$$

Let $\xi \in \mathbb{R}^n$ be a vector with nonzero components. Then the function $\vec{x} \longmapsto \phi(\langle \vec{x}, \xi\rangle)$, $\vec{x} \in \mathbb{R}^n \setminus \{0\}$, has a two-sided monogenic extension f to $S_\nu^\circ(\mathbb{R}^{n+1})$ satisfying the bounds (2.3) and the operator $f(A) \in \mathcal{L}(X)$ given by formula (2.5) has the representation

$$f(A) = \phi(\langle A, \xi\rangle) = \frac{1}{2\pi i}\int_C (\lambda I - \langle A, \xi\rangle)^{-1}\phi(\lambda)\, d\lambda. \qquad (2.7)$$

The contour C can be taken to be $\{z \in \mathbb{C} : |\operatorname{Im}(z)| = \tan\theta |\operatorname{Re}(z)|\ \}$, with $0 < \omega < \theta < \nu$.

Proof. We can suppose that $\xi \in S_0^{n-1}$ by scaling. It is routine to check that the two-sided monogenic function f is given by the Cauchy integral formula

$$f(x) = \frac{1}{2\pi i}\int_C (z - \langle \vec{x}, \xi\rangle + x_0\xi)^{-1}\phi(z)\, dz, \quad x \in S_{\nu'}^\circ(\mathbb{R}^{n+1}),$$

for all $x = x_0\mathbf{e}_0 + \vec{x} \in S^\circ_{\nu'}(\mathbb{R}^{n+1})$ and all $0 < \nu' < \theta$. Furthermore, as noted in [13, Section 5.2], the element $i\xi$ of the Clifford algebra $C\ell(\mathbb{C}^n)$ has the spectral decomposition

$$i\xi = \chi_+(\xi)|\xi| - \chi_-(\xi)|\xi|$$

with respect to the projections

$$\chi_\pm = \frac{1}{2}\left(1 \pm \frac{i\xi}{|\xi|}\right),$$

so that f also has the representation

$$f(x) = \phi(\langle\vec{x},\xi\rangle + ix_0|\xi|)\chi_+(\xi) + \phi(\langle\vec{x},\xi\rangle - ix_0|\xi|)\chi_-(\xi),$$

for all $x = x_0\mathbf{e}_0 + \vec{x} \in S^\circ_\nu(\mathbb{R}^{n+1})$. Hence, f satisfies the bounds (2.3). Note that $\langle\vec{x},\xi\rangle = 0$ if $\vec{x}$ is orthogonal to ξ, but by assumption in (2.3), $|x_0| \geq \tan\nu'|\vec{x}|$ for $\nu' > 0$.

Now suppose that $\omega' < \theta < \nu$. We first observe that

$$\int_{H_\theta} G_x(\tau A^{(k)})\mathbf{n}(x)(\zeta - \langle\vec{x},\xi\rangle + x_0\xi)^{-1}\,d\mu(x) = (\zeta I - \tau\langle A^{(k)},\xi\rangle)^{-1} \quad (2.8)$$

for every $\zeta \in \mathbb{C}\setminus S_\nu(\mathbb{C})$ and $\tau \in \mathbb{R}$. The (improper) integral is over the cone $H_\theta = \{x_0\mathbf{e}_0 + \vec{x} \in \mathbb{R}^{n+1} : |x_0| = \tan\theta|\vec{x}|\}$, which can be deformed by Cauchy's theorem in Clifford analysis to the integral over a ball B_r of radius $r > \|A^{(k)}\|$ centred at zero in $\mathbb{R}^{n+1}$ when $|\zeta|$ is large enough. Both sides of equation (2.8) are holomorphic for τ in a neighbourhood of zero in $\mathbb{C}$ and have the same Taylor series about zero, because

$$\begin{aligned}
\frac{d^j}{d\tau^j}&\int_{H_\theta} G_x(\tau A^{(k)})\mathbf{n}(x)(\zeta - \langle\vec{x},\xi\rangle + x_0\xi)^{-1}\,d\mu(x)\\
&= \int_{H_\theta} [\langle -A^{(k)},\nabla_{\vec{x}}\rangle^j G_x(\tau A^{(k)})]\mathbf{n}(x)(\zeta - \langle\vec{x},\xi\rangle + x_0\xi)^{-1}\,d\mu(x)\\
&= \int_{\mathbb{R}^n\pm\epsilon\mathbf{e}_0} \langle -A^{(k)},\nabla_{\vec{x}}\rangle^j G_x(\tau A^{(k)})\mathbf{n}(x)(\zeta - \langle\vec{x},\xi\rangle + x_0\xi)^{-1}\,d\mu(x)\\
&= \int_{\mathbb{R}^n\pm\epsilon\mathbf{e}_0} G_x(\tau A^{(k)})\mathbf{n}(x)\langle A^{(k)},\nabla_{\vec{x}}\rangle^j(\zeta - \langle\vec{x},\xi\rangle + x_0\xi)^{-1}\,d\mu(x)\\
&\qquad\longrightarrow j!\zeta^{-j-1}\langle A^{(k)},\xi\rangle^j \quad \text{as } \tau \to 0.
\end{aligned}$$

Here we have used the observation, apparent from the plane wave formula (1.2), that for each $k = 1,2,\ldots,$ the Cauchy kernel $(x,\tau) \mapsto G_x(\tau A^{(k)})$ satisfies the operator equation

$$\frac{\partial}{\partial\tau}G_x(\tau A^{(k)}) + \langle A^{(k)},\nabla_{\vec{x}}\rangle G_x(\tau A^{(k)}) = 0$$

for all $x \in N_\nu$. Then the equality (2.8) is obtained for all $\zeta \in \mathbb{C} \setminus S_\nu(\mathbb{C})$ by analytic continuation of both sides. Next, the cone H_θ can be deformed to an n-dimensional surface consisting of part of the sphere of radius $0 < r' < r$ together with $H_\theta \setminus B_{r'}$ and the contour C can be deformed so as to avoid the disk $D_{r'}$. Then it follows that the equalities

$$\begin{aligned} f(A^{(k)}) &= \int_{H_\theta} G_x(A^{(k)})\mathbf{n}(x)f(x)\,d\mu(x) \\ &= \frac{1}{2\pi i}\int_{H_\theta} G_x(A^{(k)})\mathbf{n}(x)\left[\int_C (\zeta - \langle \vec{x}, \xi\rangle + x_0\xi)^{-1}\phi(\zeta)\,d\zeta\right] d\mu(x) \\ &= \frac{1}{2\pi i}\int_C \left[\int_{H_\theta} G_x(A^{(k)})\mathbf{n}(x)(\zeta - \langle \vec{x}, \xi\rangle + x_0\xi)^{-1}\,d\mu(x)\right]\phi(\zeta)\,d\zeta \\ &= \frac{1}{2\pi i}\int_C (\zeta I - \langle A^{(k)}, \xi\rangle)^{-1}\phi(\zeta)\,d\zeta \\ &= \phi(\langle A^{(k)}, \xi\rangle) \end{aligned}$$

hold. Now the assumption (2.6) of strong resolvent convergence guarantees that $\phi(\langle A^{(k)}, \xi\rangle) \longrightarrow \phi(\langle A, \xi\rangle)$ in the strong operator topology as $k \to \infty$, so the convergence of the integrals

$$f(A^{(k)}) = \int_{H_\theta} G_x(A^{(k)})\mathbf{n}(x)f(x)\,d\mu(x)$$

to $f(A)$ as $k \to \infty$ remains to be established. According to Lemma 1, we can find $C > 0$ such that

$$\|G_x(A^{(k)})\|_{\mathcal{L}_{(n)}(X_{(n)})} \le \frac{C}{|x_0|^n}$$

for all $x \in H_\theta$ and $k = 1, 2, \ldots$. Moreover, the plane wave formula (1.2) and strong resolvent convergence (2.6) ensures that $G_x(A^{(k)}) \to G_x(A)$ in the strong operator topology for each nonzero $x \in H_\theta$. The monogenic function f has decay at zero and infinity, so dominated convergence ensures that $f(A^{(k)}) \to f(A)$ as $k \to \infty$. □

The approximation (2.6) by bounded operators is somewhat simple-minded. A more sophisticated approximation should yield the result for general systems of operators uniformly of type ω. In order to form functions $f(A)$ of the system A of operators for a class of monogenic functions f larger than those which satisfy a bound like (2.3), a greater understanding of function theory in the sector $S_\omega(\mathbb{R}^{n+1})$ is needed. To this end, the simple system $A = \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ of multiplication operators in the algebra $C\ell(\mathbb{C}^n)$ is considered in the next section.

3 Function theory in sectors

For the particular example of the n-tuple D_Σ of differentiation operators on a Lipschitz surface Σ mentioned in the Introduction, the theory described in [10] associates a bounded linear operator $f(D_\Sigma)$ with a Fourier multiplier f, this being a suitable holomorphic function of n complex variables defined in a sector in $\mathbb{C}^n$ associated with D_Σ. By means of formula (2.5), we have seen how to associate a bounded linear operator $f(D_\Sigma)$ with a left monogenic function f with suitable decay in a sector, once we verify that D_Σ satisfies the assumptions of Definition 1.

This suggests that we can associate a bounded holomorphic function of n complex variables in a sector in $\mathbb{C}^n$ with a left monogenic function f with suitable decay at zero and infinity in a sector in $\mathbb{R}^{n+1}$. The association is by analytic continuation from $\mathbb{R}^n \setminus \{0\}$ to a sector in $\mathbb{C}^n$. The precise formulation of this correspondence and its consequences is formulated in this section in Theorem 2. The proof is somewhat technical and will appear elsewhere [3]. In the next section, Theorem 2 and some additional estimates are used to extend the functional calculus defined by means of formula (2.5) to all bounded holomorphic functions defined in a sector $\mathbb{C}^n$, at least in the case when A is an n-tuple of commuting operators.

Let $0 < \nu < \frac{1}{2}\pi$ and let $H^\infty_\ell(S^\circ_\nu(\mathbb{R}^{n+1}))$ denote the set of all left monogenic functions

$$f : S^\circ_\nu(\mathbb{R}^{n+1}) \to C\ell(\mathbb{C}^n)$$

that are uniformly bounded on every subsector $S^\circ_{\nu'}(\mathbb{R}^{n+1})$, $0 < \nu' < \nu$. Endowed with the topology of uniform convergence on subsectors

$$S^\circ_{\nu'}(\mathbb{R}^{n+1}), \quad 0 < \nu' < \nu,$$

the topological vector space $H^\infty_\ell(S^\circ_\nu(\mathbb{R}^{n+1}))$ is a Fréchet space. The analogous space for right monogenic functions is written as $H^\infty_r(S^\circ_\nu(\mathbb{R}^{n+1}))$.

For any $f \in H^\infty_\ell(S^\circ_\nu(\mathbb{R}^{n+1}))$, the restriction

$$f_0 : \mathbb{R}^n \setminus \{0\} \to C\ell(\mathbb{C}^n)$$

of f to $\mathbb{R}^n \setminus \{0\}$ is real analytic and so it has a complex analytic extension $\tilde{f}$ to some open neighbourhood U_ν of $\mathbb{R}^n \setminus \{0\}$ in $\mathbb{C}^n \setminus \{0\}$. The argument above concerning Fourier multiplier operators on Lipschitz surfaces suggests that we can take U_ν to be some open sector in $\mathbb{C}^n$ on which $\tilde{f}$ is uniformly bounded on subsectors. The sectors in $\mathbb{C}^n$ are described as follows.

For $x \in \mathbb{R}^{n+1}$ and n complex variables $\zeta = (\zeta_1, \dots, \zeta_n)$, the Cauchy kernel $G_x(\zeta)$ is understood to be the maximal analytic continuation of the Cauchy kernel $\vec{y} \mapsto G_x(\vec{y})$, $\vec{y} \in \mathbb{R}^n$, $\vec{y} \neq x$, given by formula (1.4). More precisely, let

$$|x - \zeta|^2_{\mathbb{C}} = x_0^2 + \sum_{j=1}^n (x_j - \zeta_j)^2$$

and denote the positive square root of $|x-\zeta|^2_{\mathbb{C}}$ by $|x-\zeta|_{\mathbb{C}}$. In the case that x actually lies in $\{0\}\times\mathbb{R}^n\equiv\mathbb{R}^n$, the function $\zeta\mapsto|x-\zeta|_{\mathbb{C}}$ coincides with the analytic extension of the modulus function $\xi\mapsto|x-\xi|$, $\xi\in\mathbb{R}^n\setminus\{x\}$. Then

$$G_x(\zeta)=\frac{1}{\sigma_n}\frac{\overline{x}+\zeta}{|x-\zeta|^{n+1}_{\mathbb{C}}},\quad x\in\mathbb{R}^{n+1}. \tag{3.1}$$

Here we take $|x-\zeta|^2_{\mathbb{C}}\notin(-\infty,0]$ if n is an even integer and $|x-\zeta|^2_{\mathbb{C}}\neq 0$ if n is an odd integer. If the dimension n is even, then $|x-\zeta|^{n+1}_{\mathbb{C}}$ has a discontinuity as the complex number $|x-\zeta|^2_{\mathbb{C}}$ moves across $(-\infty,0]$ in $\mathbb{C}$. If n is odd, we only require the denominator $|x-\zeta|^{n+1}_{\mathbb{C}}$ to be nonzero. Then the function $(x,\zeta)\mapsto G_x(\zeta)$ is two-sided monogenic in x and holomorphic in ζ on its given domain.

According to the point of view mentioned in the Introduction, for fixed $\zeta\in\mathbb{C}^n$, the set of singularities of the Cauchy kernel $G_x(\zeta)$ as a function of $x\in\mathbb{R}^{n+1}$ is called the *monogenic spectrum* of $\zeta\in\mathbb{C}^n$ and denoted by $\gamma(\zeta)$. In fact, we are just considering $\zeta\in\mathbb{C}^n$ as an n-tuple of multiplication operators in the Clifford algebra $C\ell(\mathbb{C}^n)$ and taking the natural definition of the Cauchy kernel $x\mapsto G_x(\zeta)$. We do not need the plane wave decomposition here–the Cauchy kernel $G_x(A)$ is defined by the L^∞-functional calculus for any commuting family $A=(A_1,\dots,A_n)$ of normal operators in a Hilbert space.

To examine the subset $\gamma(\zeta)$ of $\mathbb{R}^n$ more closely, write $\zeta=\xi+i\eta$ for $\xi,\eta\in\mathbb{R}^n$ and $x=x_0\mathbf{e}_0+\vec{x}$ for $x_0\in\mathbb{R}$ and $\vec{x}\in\mathbb{R}^n$. Then

$$|x-\zeta|^2_{\mathbb{C}}=x_0^2+\sum_{j=1}^n(x_j-\zeta_j)^2=x_0^2+|\vec{x}-\xi|^2-|\eta|^2-2i\langle\vec{x}-\xi,\eta\rangle. \tag{3.2}$$

Thus, $|x-\zeta|^2_{\mathbb{C}}$ belongs to $(-\infty,0]$ if and only if x lies in the intersection hyperplane $\langle\vec{x}-\xi,\eta\rangle=0$ passing through ξ and with normal η, and the ball $x_0^2+|\vec{x}-\xi|^2\le|\eta|^2$ centred at ξ with radius $|\eta|$. If n is even, then

$$\gamma(\zeta)=\{x=x_0\mathbf{e}_0+\vec{x}\in\mathbb{R}^{n+1}:\langle\vec{x}-\xi,\eta\rangle=0,\ x_0^2+|\vec{x}-\xi|^2\le|\eta|^2\ \}. \tag{3.3}$$

and if n is odd, then

$$\gamma(\zeta)=\{x=x_0\mathbf{e}_0+\vec{x}\in\mathbb{R}^{n+1}:\langle\vec{x}-\xi,\eta\rangle=0,\ x_0^2+|\vec{x}-\xi|^2=|\eta|^2\ \}. \tag{3.4}$$

In particular, if $\mathrm{Im}(\zeta)=0$, then $\gamma(\zeta)=\{\zeta\}\subset\mathbb{R}^n$.

Now suppose that $f\in H^\infty_\ell(S^\circ_\nu(\mathbb{R}^{n+1}))$. The Cauchy integral formula gives

$$\tilde{f}(\zeta)=\int_{\partial\Omega}G_x(\zeta)\mathbf{n}(x)f(x)\,d\mu(x) \tag{3.5}$$

for a bounded open neighbourhood Ω of $\gamma(\zeta)$ with smooth oriented boundary $\partial\Omega$, outward unit normal $\mathbf{n}(x)$ at $x \in \partial\Omega$ and surface measure μ. The integral does not depend on Ω because $G_x(\zeta)$ is right monogenic in x and f is left monogenic [1, Corollary 9.3]. Differentiating under the integral sign shows that $\tilde{f}$ is holomorphic on the open subset

$$\{\gamma \in \mathbb{C}^n : \gamma(\zeta) \subset S_\nu^\circ(\mathbb{R}^{n+1})\} \quad \text{of} \quad \mathbb{C}^n,$$

which we now describe [4, Proposition 2.1].

Proposition 1. *Let $\zeta \in \mathbb{C}^n \setminus \{0\}$ and $0 < \omega < \frac{1}{2}\pi$. Then*

$$\gamma(\zeta) \subset S_\omega(\mathbb{R}^{n+1})$$

if and only if

$$|\zeta|_{\mathbb{C}}^2 \neq (-\infty, 0] \quad \text{and} \quad |\operatorname{Im}(\zeta)| \leq \operatorname{Re}(|\zeta|_{\mathbb{C}}) \tan\omega. \tag{3.6}$$

For each $0 < \omega < \frac{1}{2}\pi$, let $S_\omega(\mathbb{C}^n)$ denote the set of all $\zeta \in \mathbb{C}^n$ satisfying condition (3.6) and let $S_\omega^\circ(\mathbb{C}^n)$ be its interior. The sector $S_\omega(\mathbb{C}^n)$ arose in [10] as the set of $\zeta \in \mathbb{C}^n$ for which the exponential functions

$$e_+(x,\zeta) = e^{i\langle \vec{x},\zeta\rangle} e^{-x_0|\zeta|_{\mathbb{C}}} \chi_+(\zeta), \quad x = x_0\mathbf{e}_0 + \vec{x},$$

have a decay at infinity for all $x \in \mathbb{R}^{n+1}$ with $\langle x, m\rangle > 0$ and all unit vectors $m = m_0\mathbf{e}_0 + \vec{m} \in \mathbb{R}^{n+1}$ satisfying $m_0 \geq \cot\omega|\vec{m}|$.

For each $0 < \nu < \frac{1}{2}\pi$, let $H^\infty(S_\nu^\circ(\mathbb{C}^n))$ denote the set of all holomorphic functions

$$f : S_\nu^\circ(\mathbb{C}^n) \to C\ell(\mathbb{R}^n)$$

which are uniformly bounded on every subsector

$$S_{\nu'}^\circ(\mathbb{C}^n), \quad 0 < \nu' < \nu. \tag{3.7}$$

Endowed with the topology of uniform convergence on subsectors (3.7) the topological vector space $H^\infty(S_\nu^\circ(\mathbb{C}^n))$ is a Fréchet space and a (nonabelian) Fréchet algebra under pointwise multiplication. The subalgebra of $\mathbb{C}$-valued functions is, of course, an abelian Fréchet algebra. The proof of the following result is given in [3].

Theorem 2. *The mapping $f \mapsto \tilde{f}$ given by the Cauchy integral formula (3.5) is an isomorphism between the Fréchet spaces $H_\ell^\infty(S_\nu^\circ(\mathbb{R}^{n+1}))$ and $H^\infty(S_\nu^\circ(\mathbb{C}^n))$.*

Given two functions $f, g \in H^\infty(S_\nu^\circ(\mathbb{R}^{n+1}))$, the restrictions

$$f_0 : \mathbb{R}^n \setminus \{0\} \to C\ell(\mathbb{C}^n) \quad \text{and} \quad g_0 : \mathbb{R}^n \setminus \{0\} \to C\ell(\mathbb{C}^n)$$

of f and g to $\mathbb{R}^n \setminus \{0\}$ are real analytic, so their product $f_0 g_0$ has a left monogenic extension $f \cdot_\ell g$ to an open neighbourhood of $\mathbb{R}^n \setminus \{0\}$ in $\mathbb{R}^{n+1}$. The analogous product for right monogenic functions is written as $f \cdot_r g$.

Corollary 1. *For each $f, g \in H^\infty_\ell(S^\circ_\nu(\mathbb{R}^{n+1}))$, the function $f \cdot_\ell g$ has a left monogenic extension to $S^\circ_\nu(\mathbb{R}^{n+1})$, denoted by the same symbol, and $f \cdot_\ell g \in H^\infty_\ell(S^\circ_\nu(\mathbb{R}^{n+1}))$. Moreover, $H^\infty_\ell(S^\circ_\nu(\mathbb{R}^{n+1}))$ is a Fréchet algebra with the product*

$$(f, g) \mapsto f \cdot_\ell g.$$

The mapping $f \mapsto \tilde{f}$ given by the Cauchy integral formula (3.5) is an isomorphism of the Fréchet algebras $H^\infty_\ell(S^\circ_\nu(\mathbb{R}^{n+1}))$ and $H^\infty(S^\circ_\nu(\mathbb{C}^n))$.

If the restrictions f_0, g_0 of f and g take values in $\mathbb{C}$ rather than $C\ell(\mathbb{R}^n)$, then both f and g are two-sided monogenic and

$$\widetilde{f \cdot_r g} = \widetilde{f \cdot_\ell g} = \tilde{f}.\tilde{g} = \tilde{g}.\tilde{f}.$$

Hence, $f \cdot_\ell g = g \cdot_\ell f$ and the subalgebra of $H^\infty_\ell(S^\circ_\nu(\mathbb{R}^{n+1}))$ consisting of such functions is abelian.

4 Joint spectral theory of operators of type ω

Suppose that $T : \mathcal{D}(T) \to \mathcal{H}$ is a single densely defined linear operator acting in the Hilbert space $\mathcal{H}$. If $0 \le \omega < \frac{1}{2}\pi$ is a number for which T is of type ω (see Definition 1), then the one–dimensional version of formula (2.5) gives a bounded linear operator $f(A)$ defined by

$$f(T) = \frac{1}{2\pi i}\int_C (\lambda I - T)^{-1} f(\lambda)\, d\lambda \tag{4.1}$$

for any function f satisfying the bounds (2.3) in $S_\nu(\mathbb{C})$ in the case $n = 1$. The contour C can be taken to be

$$\{z \in \mathbb{C} : |\operatorname{Im}(z)| = \tan\theta |\operatorname{Re}(z)|\ \},$$

with $\omega < \theta < \nu$. The operator T of type ω is said to have a *bounded H^∞-functional calculus* if for each $\omega < \nu < \frac{1}{2}\pi$, there exists an algebra homomorphism $f \longmapsto f(T)$ from $H^\infty(S^\circ_\nu(\mathbb{C}))$ to $\mathcal{L}(\mathcal{H})$ agreeing with (4.1) and a positive number C_ν such that

$$\|f(T)\| \le C_\nu \|f\|_\infty \quad \text{for all} \quad f \in H^\infty(S^\circ_\nu(\mathbb{C})).$$

The following result is from [11, Theorem 6.2.2]

Theorem 3. *Suppose that T is a one-to-one operator of type ω in $\mathcal{H}$. Then T has a bounded H^∞-functional calculus if and only if for every $\omega < \nu < \frac{1}{2}\pi$, there exists $c_\nu > 0$ such that T and its adjoint T^* satisfy the square function estimates*

$$\int_0^\infty \|\psi_t(T)u\|^2 \frac{dt}{t} \le c_\nu \|u\|^2, \quad u \in \mathcal{H}, \tag{4.2}$$

$$\int_0^\infty \|\psi_t(T^*)u\|^2 \frac{dt}{t} \le c_\nu \|u\|^2, \quad u \in \mathcal{H}, \tag{4.3}$$

for some function $\psi \in H^\infty(S^\circ_\nu(\mathbb{C}))$, *which satisfies*

$$\int_0^\infty \psi^3(t)\,\frac{dt}{t} = \int_0^\infty \psi^3(-t)\,\frac{dt}{t} = 1, \text{ and} \tag{4.4}$$

$$|\psi(z)| \le C_\nu \frac{|z|^s}{1+|z|^{2s}}, \quad z \in S^\circ_\nu(\mathbb{C}), \tag{4.5}$$

for some $s > 0$. *Here* $\psi_t(z) = \psi(tz)$ *for* $z \in S^\circ_\nu(\mathbb{C})$.

We now use formula (2.5) to generalise this result to n-tuples of commuting operators acting in a Hilbert space $\mathcal{H}$.

Theorem 4. *Let* $A = (A_1, \dots, A_n)$ *be an* n*-tuple of densely defined commuting linear operators* $A_j : \mathcal{D}(A_j) \to \mathcal{H}$ *acting in a Hilbert space* $\mathcal{H}$ *such that* $\cap_{j=1}^n \mathcal{D}(A_j)$ *is dense in* $\mathcal{H}$. *Suppose that* $0 \le \omega < \frac{1}{2}\pi$ *and* A *is uniformly of type* ω.

If $T = i(A_1\mathbf{e}_1 + \cdots + A_n\mathbf{e}_n)$ *is a one-to-one operator of type* ω *acting in* $\mathcal{H}_{(n)}$ *and* T *has an* H^∞*-functional calculus, then the* n*-tuple* A *has a bounded* H^∞*-functional calculus on* $S^\circ_\nu(\mathbb{C}^n)$ *for any* $\omega < \nu < \frac{1}{2}\pi$, *that is, there exists a homomorphism* $b \longmapsto b(A)$, $b \in H^\infty(S^\circ_\nu(\mathbb{C}^n))$, *from* $H^\infty(S^\circ_\nu(\mathbb{C}^n))$ *into* $\mathcal{L}_{(n)}(\mathcal{H}_{(n)})$ *and there exists* $C_\nu > 0$ *such that*

$$\|b(A)\| \le C_\nu \|b\|_\infty \quad \text{for all} \quad b \in H^\infty(S^\circ_\nu(\mathbb{C}^n)).$$

Moreover, if f *is the unique two-sided monogenic function defined on* $S^\circ_\nu(\mathbb{R}^{n+1})$ *such that* $b = \tilde{f}$, *as in Theorem 2, and* f *satisfies the bound (2.3), then* $b(A) = f(A)$ *is given by formula (2.5).*

Proof. By assumption, the operator T has an H^∞-functional calculus, so there exists a function $\psi \in H^\infty(S^\circ_\nu(\mathbb{C}))$ satisfying conditions (4.2) – (4.5). Our aim is to define $b(A)$ for $b \in H^\infty(S^\circ_\nu(\mathbb{C}^n))$, by the formula

$$(b(A)u, v) = \int_0^\infty \Big((b\phi_t)(A)\psi_t(T)u, \psi_t(T)^* v \Big) \frac{dt}{t} \tag{4.6}$$

for all $u, v \in \mathcal{H}_{(n)}$. The function $\phi : S^\circ_\nu(\mathbb{C}^n) \to \mathbb{C}$ is constructed from ψ by setting

$$\phi(\zeta) = \psi^2\{i\zeta\} = \psi^2(|\zeta|_{\mathbb{C}})\chi_+(\zeta) + \psi^2(-|\zeta|_{\mathbb{C}})\chi_-(\zeta),$$

for all $\zeta \in S^\circ_\nu(\mathbb{C}^n)$. Then for a particular choice of the function ψ [3], the holomorphic function ϕ_t defined for $t > 0$ by $\phi_t(\zeta) = \phi(t\zeta)$ has the property that

$$\vec{x} \mapsto b(\vec{x})\phi_t(\vec{x}), \quad \vec{x} \in \mathbb{R}^n \setminus \{0\}, \tag{4.7}$$

has a left (and right) monogenic extension $b \cdot_\ell \phi_t$ to $S^\circ_\nu(\mathbb{R}^{n+1})$ with decay at zero and infinity [3]. Hence $(b\phi_t)(A) := (b \cdot_\ell \phi_t)(A)$ is defined by formula (2.5) and satisfies

$$\sup_{t>0} \|(b \cdot_\ell \phi_t)(A)\| \le C\|b\|_\infty \,.$$

By normalising ψ so that

$$\int_0^\infty \psi^4(t)\,\frac{dt}{t} = 1,$$

the desired functional calculus $b \mapsto b(A)$, $b \in H^\infty(S^\circ_\nu(\mathbb{C}^n))$, is obtained. □

Remark. (i) In [13], it is shown that the n-tuple D_Σ of differentiation operators on a Lipschitz surface $\Sigma \subset \mathbb{R}^{n+1}$ mentioned in the Introduction satisfies the conditions of Theorem 4. In particular $\langle D_\Sigma, s\rangle$ is the differentiation operator on the Lipschitz graph determined by the slice of Σ in the direction $s \in S^{n-1}$, so for some $0 \le \omega < \frac{1}{2}\pi$, the n-tuple D_Σ is uniformly of type ω.

(ii) A result similar to Theorem 4 is obtained in [11, Theorem 6.4.3] using the idea that A generates a bounded monogenic semigroup rather than the assumption that A is uniformly of type ω. I do not know the connection between the two concepts.

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Brian Jefferies
School of Mathematics
UNSW
Kensington
NSW 2052, Australia
E-mail: b.jefferies@unsw.edu.au

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