MATH 1231 Mathematics 1B Algebra S2 2008 Test 2 Version 2B (Yellow)

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1. (a)

$$T\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ -3(x_1+y_1)+(x_2+y_2) \\ 2(x_2+y_2) \end{pmatrix}$$
$$= \begin{pmatrix} x_1 \\ -3x_1+x_2 \\ 2x_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ -3x_1+x_2 \\ 2x_2 \end{pmatrix}$$
$$= T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + T\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$
$$T\left(\lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} \lambda x_1 \\ -3\lambda x_1+\lambda x_2 \\ 2\lambda x_2 \end{pmatrix}$$
$$= \lambda\begin{pmatrix} x_1 \\ -3x_1+x_2 \\ 2x_2 \end{pmatrix}$$
$$= \lambda T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(b) $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and so the matrix is

$$A_T = \begin{pmatrix} 1 & 0 \\ -3 & 1 \\ 0 & 2 \end{pmatrix}$$

2.

$$\begin{pmatrix} -1 & 3 & 1 & 3 & 0 \\ 2 & -5 & -1 & -4 & -1 \\ -2 & 7 & 3 & 8 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 3 & 1 & 3 & 0 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

rank $A = =$ no. of leading columns $= 2$

nullity $A =$ no. of non-leading columns $= 3$

Bases for the image $=$ first two columns of $A = \left\{ \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix} \right\}$.

3. $\det \begin{pmatrix} 5-\lambda & 3 \\ 2 & 6-\lambda \end{pmatrix} = (5-\lambda)(6-\lambda) - 6 = 0 \Rightarrow \lambda = 8, 3$. We need to find the kernel of $A - 8I$ and $A - 3I$.

$$\begin{aligned} A - 8I &= \begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix} \\ &\to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

So $\lambda = 8$ is an eigenvalue with eigenvectors $t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $t \neq 0$.

$$\begin{aligned} A - 3I &= \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \\ &\to \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

So $\lambda = 3$ is an eigenvalue with eigenvector $t\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ for $t \neq 0$.