MATH 1231 Mathematics 1B Algebra S2 2008 Version 2A (Blue)

Boris Lerner

September 2, 2008

1. Since we need to show something is not a subspace all we have to do is to show one of the 10 axioms of being a vector space is violated. It is clear that

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix} \in S$$

but

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \notin S.$$

Thus S is not closed under vector addition and hence is not a subspace of $\mathbb{R}^2$.

2. (a) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ can not possibly span $\mathbb{R}^2$ since the span has at most dimension 3 where as $\mathbb{R}^4$ has dimension 4.

For $\mathbf{b}$ to be in the span there must exists $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 = \mathbf{b}.$$

Putting in matrix form we get

$$\left(\begin{array}{rrr|l} 1 & -1 & -2 & b_1 \\ -1 & 2 & 3 & b_2 \\ -2 & 3 & 5 & b_3 \\ 2 & 1 & -1 & b_4 \end{array}\right)$$

After row reducing (not shown) we get

$$\left(\begin{array}{rrr|l} 1 & -1 & -2 & b_1 \\ 0 & 1 & 1 & b_2 + b_3 \\ 0 & 0 & 0 & b_1 - b_2 + b_3 \\ 0 & 0 & 0 & -5b_1 - 3b_2 + b_4 \end{array}\right)$$

and so the condition for $\mathbf{b}$ to be in the span is

$$b_1 - b_2 + b_3 = 0 \quad \text{and} \quad -5b_1 - 3b_2 + b_4 = 0$$

(b) We need to check $\mathbf{a}$ satisfies the two conditions from (a). $1 - 2 + 1 = 0$ so satisfies the first condition, and $-5 \times 1 - 3 \times 2 + 6 \neq 0$ so doesn't satisfy the second condition and so $\mathbf{a}$ is not in the span.

3. (a) Note first of all that this is obvious since any 4 vectors in the a 3 dimensional space have to be linearly dependent. Now for a formal proof: we need to show that the equation

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 + \lambda_4 \mathbf{v}_4 = \mathbf{0}$$

has infinitely many solutions. We form the corresponding augmented matrix

$$\left(\begin{array}{rrrr|r} -1 & 2 & -1 & -2 & 0 \\ -3 & 5 & 0 & -5 & 0 \\ 2 & -5 & 4 & 3 & 0 \end{array}\right)$$

and upon row reducing we get

$$\left(\begin{array}{rrrr|r} -1 & 2 & -1 & -2 & 0 \\ 0 & -1 & 3 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 \end{array}\right)$$

and since the second last column is not leading, we see that the system has infinitely many solutions.

(b) From (i) we can in fact see that since the second last column corresponds to $\mathbf{v}_4$ we should be able to write $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. That is, we want to solve

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 = \mathbf{v}_4.$$

We form the corresponding matrix (note the similarity to (i))

$$\left(\begin{array}{rrr|r} -1 & 2 & -1 & -2 \\ -3 & 5 & 0 & -5 \\ 2 & -5 & 4 & 3 \end{array}\right)$$

and upon rod reducing we get

$$\left(\begin{array}{rrr|l} -1 & 2 & -1 & -2 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -1 & -2 \end{array}\right)$$

Back substituting,we get $\lambda_3 = 2, \lambda_2 = 5, \lambda_1 = 10$ and so

$$10\mathbf{v}_1 + 5\mathbf{v}_2 + 2\mathbf{v}_3 = \mathbf{v}_4$$