MATH 1251 Mathematics 1B Algebra S2 2008 Version 3A (Blue)

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1. We solve $(x+iy)^2 = 40 - 42i$. After expanding and equating real and imaginary parts and modulii we get:

$$\begin{aligned} x^2 - y^2 &= 40 \quad (1) \\ 2xy &= -42 \quad (2) \\ x^2 + y^2 &= 58 \quad (3) \end{aligned}$$

Granted, this is probably not so easy without a calculator. From this is easy to deduce $x + iy = \pm(7 - 3i)$. I was lenient when marking.

2. This was the hard question of the test. Note that Daniel Chan did go through a similar one in lectures. We have:

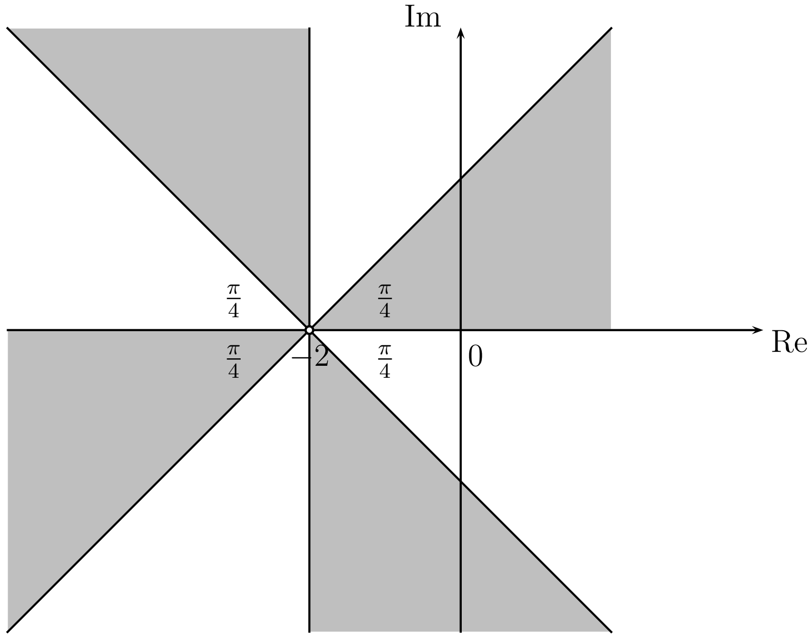
$$0 \leq \text{Arg}\left((z+2)^4\right) \leq \pi$$

This implies

$$\begin{aligned} & 0 \leq 4\text{Arg}\,(z+2) + 2k\pi \leq \pi \\ \Rightarrow \quad & -\frac{k\pi}{2} \leq \text{Arg}\,(z+2) \leq \frac{\pi - 2k\pi}{4} \end{aligned}$$

For $k = -1, 0, 1, 2$ this gives

$$\begin{aligned} 0 \leq \text{Arg}\,(z+2) &\leq \frac{\pi}{2} \\ \frac{\pi}{2} \leq \text{Arg}\,(z+2) &\leq \frac{3\pi}{4} \\ -\frac{\pi}{2} \leq \text{Arg}\,(z+2) &\leq -\frac{\pi}{4} \\ -\pi \leq \text{Arg}\,(z+2) &\leq -\frac{3\pi}{4} \end{aligned}$$



3. (a) The characteristic equation is $3\lambda^2 + 4\lambda + 2 = 0$ and so

$$\lambda = -\frac{2}{3} \pm \frac{i\sqrt{2}}{3}.$$

Since $|\lambda| = \sqrt{\frac{2}{3}} < 1$ the system is stable.

(b) Since Re(λ)$= -\frac{2}{3} < 0$ the solution is stable.

4. (a) α is a root of $x^5 - 0$. Further $x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1)$ and since $\alpha \neq 1$ we must have $\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1$.

(b)

$$\begin{aligned}\left(z - \alpha - \alpha^{-1}\right)\left(z - \alpha^2 - \alpha^{-2}\right) &= z^2 - z\left(\alpha^2 + \alpha + \alpha^{-1} + \alpha^{-2}\right) \\ &\quad + \left(\alpha^3 + \alpha + \alpha^{-1} + \alpha^{-3}\right) \\ &= z^2 + z - 1\end{aligned}$$

since $\alpha^{-1} = \alpha^4, \alpha^{-2} = \alpha^3, \alpha^{-3} = \alpha^2$.

5.

$$\begin{aligned}
&\Rightarrow \mathrm{Im}\left(\frac{4+3i}{a+2i}\right)=10\\
&\Rightarrow \mathrm{Im}\left(\frac{(4+3i)(a-2i)}{a^2+4}\right)=10\\
&\Rightarrow 3a-8=10a^2+40\\
&\Rightarrow 5a^2-15a+24=0
\end{aligned}$$

which (unfortunatelly??) has no solutions.

6. We need to solve $z^6=-64i=64e^{2k\pi+\pi}$. Let $z=re^{i\theta}$ and so $r^6=64$ and so $r=2$. Also, $4\theta=2k\pi+\pi$ and so $\theta=\frac{2k\pi+\pi}{4}$. Thus

$$z=2e^{\frac{2k\pi+\pi}{4}} \quad ,k=-2,-1,0,1.$$

So $z=2e^{-\frac{3i\pi}{4}},2e^{-\frac{i\pi}{4}},2e^{\frac{i\pi}{4}},2e^{\frac{3i\pi}{4}}$.

7. This was the hard question of the test. Note that Daniel Chan did go through a similar one in lectures. We have:

$$0\le \mathrm{Arg}\left((z-1-3i)^3\right)\le\frac{3\pi}{4}$$

This implies

$$\begin{aligned}
&0\le 3\mathrm{Arg}\,(z-1-3i)+2k\pi\le\frac{3\pi}{4}\\
\Rightarrow\quad &-\frac{2k\pi}{3}\le \mathrm{Arg}\,(z-1-3i)\le\frac{3\pi-8k\pi}{12}
\end{aligned}$$

For $k=-1,0,1$ this gives

$$\begin{aligned}
0\le \mathrm{Arg}\,(z-1-3i)&\le\frac{\pi}{4}\\
-\frac{2\pi}{3}\le \mathrm{Arg}\,(z-1-3i)&\le\frac{-5\pi}{12}\\
\frac{2\pi}{3}\le \mathrm{Arg}\,(z-1-3i)&\le\frac{11\pi}{12}
\end{aligned}$$

Hopefully, you can sketch these regions. Come see me if you want.

8. Its a GP with common ratio of $e^{2i\theta}$ and there are n terms. Thus the sum is