MATH 1251 Mathematics 1B Algebra S2 2008 Version 2A (Blue)

Boris Lerner

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1. Use the quadratic formula to get:

$$z = \frac{4 - i \pm \sqrt{(4-i)^2 - 4(5+i)}}{2} = \frac{4 - i \pm \sqrt{-5-12i}}{2}.$$

Let $(x+iy)^2 = -5 - 12i$. We get

$$\begin{aligned} x^2 - y^2 &= -5 && (1) \\ 2xy &= -12 && (2) \\ x^2 + y^2 &= 13 && (3) \end{aligned}$$

This gives $x + iy = \pm(2 - 3i)$. Thus

$$z = \frac{4 - i \pm (2-3i)}{2}$$

and so $z = 1 + i, 3 - 2i$.

2. Let $z = re^{i\theta}$ such that $z^4 = 8 + 8\sqrt{3}i$. Then, since $z^4 = r^4 e^{4i\theta}$ we have $r^4 = |8 + 8\sqrt{3}i| = 16 \Rightarrow r = 2$. Also, $4\theta = \arg\,(8 + 8\sqrt{3}i) \Rightarrow 4\theta = \frac{\pi}{3} + 2k\pi$ and so

$$\theta = \frac{\pi + 6k\pi}{12}, \quad k = -2, -1, 0, 1.$$

Thus

$$z = 2e^{-\frac{11\pi i}{12}}, 2e^{-\frac{5\pi i}{12}}, 2e^{\frac{\pi i}{12}}, 2e^{\frac{7\pi i}{12}}.$$

3. This was the hard question of the test. Note that Daniel Chan did go through a similar one in lectures. We have:

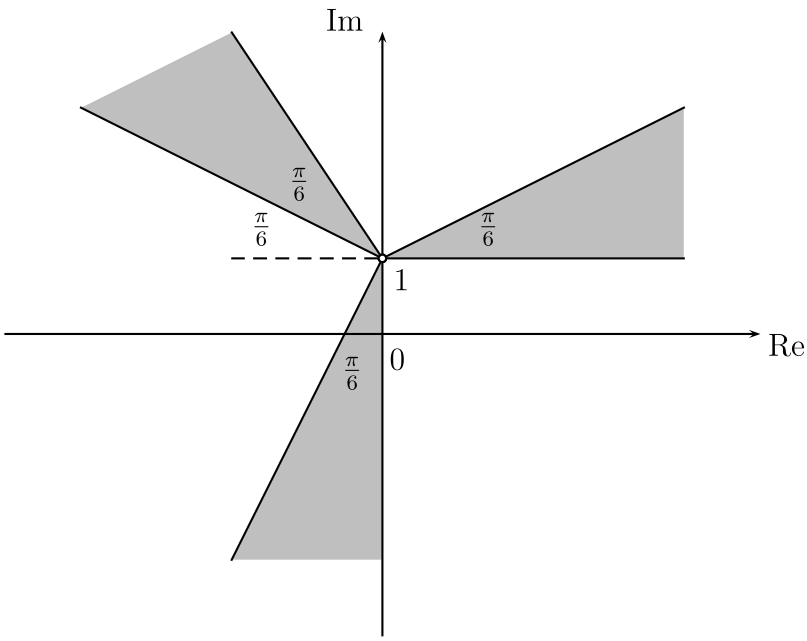
$$0 \le \text{Arg}\left((z-i)^3\right) \le \frac{\pi}{2}$$

This implies

$$0 \leq 3\mathrm{Arg}\,(z-i) + 2k\pi \leq \frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{2k\pi}{3} \leq \mathrm{Arg}\,(z-i) \leq \frac{\pi - 4k\pi}{6}$$

For $k = -1, 0, 1$ this gives $0 \leq \mathrm{Arg}\,(z-i) \leq \frac{\pi}{6}$, $-\frac{2\pi}{3} \leq \mathrm{Arg}\,(z-i) \leq -\frac{\pi}{2}$, $\frac{2\pi}{3} \leq \mathrm{Arg}\,(z-i) \leq \frac{5\pi}{6}$.



4. (a) We form the characteristic equation

$$5\lambda^2 - 2\lambda + 4 = 0$$

and solve it to get

$$\lambda = \frac{1}{5}\left(1 \pm i\sqrt{19}\right).$$

Thus $|\lambda| = \frac{\sqrt{20}}{5} < 1$ and so the solution is stable.

(b) Same as above only here we are interested in $\mathrm{Re}(\lambda)$. Since $\mathrm{Re}(\lambda) = \frac{1}{5} > 0$ the solution is not stable.