COVARIANT REPRESENTATIONS OF HECKE ALGEBRAS AND IMPRIMITIVITY FOR CROSSED PRODUCTS BY HOMOGENEOUS SPACES

ASTRID AN HUEF, S. KALISZEWSKI, AND IAIN RAEBURN

Abstract. For discrete Hecke pairs (G, H), we introduce a notion of covariant representation which reduces in the case where H is normal to the usual definition of covariance for the action of G/H on $c_0(G/H)$ by right translation; in many cases where G is a semidirect product, it can also be expressed in terms of covariance for a semigroup action. We use this covariance to characterise the representations of $c_0(G/H)$ which are multiples of the multiplication representation on $\ell^2(G/H)$, and more generally, we prove an imprimitivity theorem for regular representations of certain crossed products by coactions of homogeneous spaces. We thus obtain new criteria for extending unitary representations from H to G.

Introduction

Let G be a locally compact group, and let H be a closed subgroup of G. We have recently ([14, 12, 13]) been working on the problem of extending unitary representations from H to G, using the theory of non-abelian crossed-product duality. Our techniques reduce the extension problem to one of *imprimitivity*; that is, to deciding whether a certain induced representation is equivalent to a particular type of regular representation. At its core, the latter involves characterising the representations of $C_0(G/H)$ which are equivalent to a multiple $1 \otimes M$ of the representation M by multiplication on $L^2(G/H)$.

If H is normal in G, such a characterisation can be obtained from the Stone-von Neumann theorem: a given representation ν of $C_0(G/H)$ is equivalent to a multiple of M if and only if there exists a unitary representation U of G/H such that the pair (ν, U) is covariant for the action rt of G/H on $C_0(G/H)$ by right translation.

Motivated by the desire to extend our techniques to the non-normal case, in this paper we obtain a similar characterisation which works when H is a Hecke subgroup of a discrete group G. With the Hecke algebra $\mathcal{H}(G, H)$ playing the role of the group G/H, we formulate a covariance condition for pairs (ν, V) of representations of $c_0(G/H)$ and $\mathcal{H}(G, H)$, and use it to prove a Stone-von Neumann-type theorem characterising the representations of $c_0(G/H)$ which are equivalent to a multiple of M. We then use our theorem to solve the imprimitivity problem mentioned above, and we apply this to obtain new results on the extension problem for representations.

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Since the Hecke algebra does not act on $c_0(G/H)$ in any obvious way, our covariance condition (Definition 1.1) looks a little unusual. However, when H is normal in G, representations of $\mathcal{H}(G,H)$ correspond to unitary representations of G/H, and under this correspondence the condition reduces to the more familiar covariance condition for representations of $(c_0(G/H), G/H, \mathrm{rt})$ mentioned above (see Remark 1.2). Moreover, in a large class of examples where $\mathcal{H}(G,H)$ can be realised as a semigroup crossed product, our covariance condition can be expressed in terms of the existing notions for group and semigroup actions (Theorem 2.1), and is also closely related to recent work of Exel [10] and Larsen [19] (see Proposition 2.3).

In somewhat more detail, our first main result (Theorem 1.6) states that the covariant representations of $(c_0(G/H), \mathcal{H}(G,H))$ are precisely those pairs which are equivalent to a multiple of (M, ρ), where ρ is the natural representation of $\mathcal{H}(G,H)$ on $\ell^2(G/H)$ analogous to the right-regular representation of G/H. It follows easily that a given representation ν of $c_0(G/H)$ is equivalent to $1 \otimes M$ if and only if there exists a representation V of $\mathcal{H}(G,H)$ such that (ν, V) is a covariant pair. Since G is discrete, our proof is more elementary than that of the Stone-von Neumann theorem, although it is based on the same observation: covariant pairs generate sets of operators which behave like matrix units.

In Section 3, we use Theorem 1.6 to obtain a new imprimitivity theorem for C^*-crossed products by maximal coactions. (This is the natural abstract setting for our application to the extension problem.) Since G is discrete, a coaction δ of G on a C^*-algebra B is best viewed as a *Fell bundle* over G, which is an analytic version of a grading of B by G: for each $x \in G$, the set $B_x = \{b \in B \mid \delta(b) = b \otimes x\}$ is a linear subspace of B, we have $B_x B_y \subseteq B_{xy}$ and $B_x^* = B_{x^{-1}}$ for $x, y \in G$, and $\cup_{x\in G} B_x$ spans a dense subspace of B ([21]). For a subgroup H of G, Echterhoff and Quigg ([9]) have defined a crossed product C^*-algebra $B \times_{\delta|} (G/H)$ which, if δ is maximal, is universal for suitably covariant pairs of representations of B and $c_0(G/H)$. By definition, the *regular representations* of this crossed product are those induced from a representation θ of B via the covariant pair $((\theta \otimes \lambda) \circ \delta, 1 \otimes M)$, where λ is the quasi-regular representation of G on $\ell^2(G/H)$. Our imprimitivity theorem (Theorem 3.2) characterises these regular representations up to unitary equivalence as those pairs (π, ν) for which there is a representation V of $\mathcal{H}(G,H)$ in the commutant of π such that (ν, V) is a covariant pair.

When we apply Theorem 3.2 to the extension problem, we obtain a general result (Theorem 4.3) for representations of C^*-dynamical systems involving actions of G. To see what this says about group representations, recall that the group C^*-algebra $C^*(H)$ is naturally Morita equivalent to the crossed product $C^*(G) \times_{\delta_G|} (G/H)$, where δ_G is the comultiplication on $C^*(G)$, and thus there is a bijective correspondence $U \mapsto (\pi_U, \nu_U)$ between unitary representations of H and covariant representations of $(C^*(G), c_0(G/H))$. Theorem 1 of [12] says that U extends to a representation of G if and only if the representation of $C^*(G) \times_{\delta_G|} (G/H)$ corresponding to (π_U, ν_U) is equivalent to a regular representation. (The term "coaction-regular" was used in [12].) Thus, by Theorem 3.2, U extends if and only if there is a representation V of $\mathcal{H}(G,H)$ in the commutant of π_U such that (ν_U, V) is a covariant pair.

Conventions. Let (G,H) be a discrete Hecke pair: this means that H is a subgroup of a discrete group G such that every double coset HxH contains just finitely many left cosets. We use R to denote the right coset counting map, so that

$$R(x) = |H\backslash HxH| = |Hx^{-1}H/H| < \infty$$

for all $x \in G$. We view the Hecke algebra $\mathcal{H}(G, H)$ as the $*$-algebra of finitely-supported functions on the double-coset space $H\backslash G/H$ with the operations given for $HxH \in H\backslash G/H$ by

$$fg(HxH) = \sum_{yH \in G/H} f(HyH)g(Hy^{-1}xH) \qquad \text{and} \qquad f^*(HxH) = \overline{f(Hx^{-1}H)}.$$

We write $[HxH]$ for the characteristic function of the double coset HxH, viewed as an element of $\mathcal{H}(G, H)$, even when HxH happens to be a single left or right coset. We use ϵ_{xH} to denote the characteristic function of the coset xH, viewed as an element of $\ell^2(G/H)$ or $c_0(G/H)$, and we use χ to denote characteristic functions in other contexts.

All representations of C^*-algebras appearing in this paper are implicitly nondegenerate $*$-homomorphisms. All representations of Hecke algebras are unital $*$-representations, and we will often re-state this explicitly for emphasis.

1. Covariant representations

To understand where our new covariance condition comes from, we re-examine the group case in more detail. Let H be a normal subgroup of a locally compact group G, and let ρ be the right-regular representation of G/H on $L^2(G/H)$. Then the Stone-von Neumann theorem says that the crossed product $C_0(G/H) \times_{\mathrm{rt}} (G/H)$ is isomorphic to the algebra $\mathcal{K}(L^2(G/H))$ of compact operators via the integrated form $M \times \rho$ of the covariant representation (M, ρ) (see, for example, [22, Theorem C.34]). Since every representation of the compacts is equivalent to a multiple of the identity representation, one can deduce that every covariant representation (ν, U) of $(C_0(G/H), G/H, \mathrm{rt})$ is equivalent to a multiple of (M, ρ).

For a discrete group G, however, the same conclusion follows from the more elementary observation that the operators $\nu(\epsilon_{xH})U(x^{-1}yH)\nu(\epsilon_{yH})$ generate a set of matrix units as xH and yH run through G/H. It is this approach that we will extend to Hecke subgroups.

Definition 1.1. Let (G, H) be a discrete Hecke pair, let ν be a nondegenerate $*$-representation of $c_0(G/H)$ on a Hilbert space $\mathcal{H}$, and let V be a unital $*$-representation of the Hecke algebra $\mathcal{H}(G, H)$ on the same Hilbert space.

We say that (ν, V) is a *matrix unit pair* if the collection

$$\{\nu(\epsilon_{xH})V([Hx^{-1}yH])\nu(\epsilon_{yH}) \mid xH, yH \in G/H\}$$

is a set of matrix units in $B(\mathcal{H})$.

We say that (ν, V) is a *covariant pair* if

$$V([HaH])\nu(\epsilon_{xH})V([HbH]) = \sum_{\substack{uH \subseteq Ha^{-1}H \\ vH \subseteq HbH}} \nu(\epsilon_{xuH})V([Hu^{-1}vH])\nu(\epsilon_{xvH}) \tag{1.1}$$

for all $a, x, b \in G$.

Note that both properties in Definition 1.1 are preserved by unitary equivalence.

Remark 1.2. When H is normal in G, the Hecke algebra is just the group algebra $\mathbb{C}(G/H)$, and we can convert between unital $*$-representations of $\mathbb{C}(G/H)$ and unitary representations of G/H by identifying group elements in G/H with their characteristic functions. It follows that a pair (ν, V) is covariant for (G, H) if and only if it is covariant in the usual sense for the action rt of G/H on $c_0(G/H)$ by right translation. Indeed, in this case the sums in (1.1) disappear and we get

$$V(aH)\nu(\epsilon_{xH})V(bH) = \nu(\epsilon_{xa^{-1}H})V(abH)\nu(\epsilon_{xbH}) \tag{1.2}$$

for all $a, x, b \in G$. Taking $b = a^{-1}$, we recover the usual covariance condition:

$$V(aH)\nu(\epsilon_{xH})V(aH)^* = \nu(\epsilon_{xa^{-1}H}) = \nu(\mathrm{rt}_{aH}(\epsilon_{xH})). \tag{1.3}$$

Conversely, condition (1.2) can be derived from (1.3) by writing

$$\begin{aligned} V(aH)\nu(\epsilon_{xH})V(bH) &= V(aH)\nu(\epsilon_{xH})V(aH)^*V(abH)V(b^{-1}H)\nu(\epsilon_{xH})V(b^{-1}H)^* \\ &= \nu(\mathrm{rt}_{aH}(\epsilon_{xH}))V(abH)\nu(\mathrm{rt}_{b^{-1}}(\epsilon_{xH})) \\ &= \nu(\epsilon_{xa^{-1}H})V(abH)\nu(\epsilon_{xbH}). \end{aligned}$$

It will follow from Proposition 1.4 that every covariant pair is a matrix unit pair; we do not know if the converse is true in general (see also Remark 1.7). In the group case, however, the covariant pairs are exactly the matrix unit pairs, since then condition (iv) of Lemma 1.3 is satisfied by every matrix unit pair.

Lemma 1.3. *Let (G, H) be a discrete Hecke pair, let ν be a nondegenerate $*$-representation of $c_0(G/H)$, and let V be a unital $*$-representation of $\mathcal{H}(G, H)$ on the same Hilbert space $\mathcal{H}$. Then the following are equivalent*:

(i) $V([HaH])\nu(\epsilon_{xH}) = \sum_{uH\subseteq Ha^{-1}H} \nu(\epsilon_{xuH})V([HaH])\nu(\epsilon_{xH})$ *for all $a, x \in G$.*

(ii) $\nu(\epsilon_{xH})V([HbH]) = \sum_{vH\subseteq HbH} \nu(\epsilon_{xH})V([HbH])\nu(\epsilon_{xvH})$ *for all $x, b \in G$.*

(iii) $\nu(\epsilon_{xH})V([HbH])\nu(\epsilon_{yH}) = 0$ *unless $Hx^{-1}yH = HbH$.*

If (ν, V) is a matrix unit pair, then (i)–(iii) *are also equivalent to*:

(iv) $\|V([HaH])|_{\nu(\epsilon_{xH})\mathcal{H}}\| \leq R(a)^{1/2}$ *for all $a, x \in G$, where R is the right-coset counting map.*

Proof. The equivalence of (i) and (ii) is easily seen on taking adjoints. Suppose condition (ii) holds. Then for any $a, x, b \in G$,

$$\nu(\epsilon_{xH})V([HbH])\nu(\epsilon_{yH}) = \sum_{vH\subseteq HbH} \nu(\epsilon_{xH})V([HbH])\nu(\epsilon_{xvH})\nu(\epsilon_{yH}) = 0$$

unless $xtH = yH$ for some $tH \subseteq HbH$, which is precisely when $x^{-1}yH \subseteq HbH$, which is precisely when $Hx^{-1}yH = HbH$. Thus (ii) implies (iii).

Next, assume condition (iii). Since ν is nondegenerate, to establish (ii) it suffices to show that

$$\nu(\epsilon_{xH})V([HbH])\nu(\epsilon_{yH})h = \sum_{vH\subseteq HbH} \nu(\epsilon_{xH})V([HbH])\nu(\epsilon_{xvH})\nu(\epsilon_{yH})h$$

for each $y \in G$ and $h \in \mathcal{H}$. By assumption, the left-hand side is zero unless $Hx^{-1}yH = HbH$; the right-hand side is zero unless $xvH = yH$ for some $vH \subseteq HbH$, which is precisely when $Hx^{-1}yH = HbH$. When $Hx^{-1}yH \neq HbH$, the sum on the right collapses to the single term on the left. Thus (iii) implies (ii).

Now suppose (ν, V) is a matrix unit pair such that (i) holds, and fix $a, x \in G$. Then for each $uH \subseteq Ha^{-1}H$, the matrix unit $\nu(\epsilon_{xuH})V([H(xu)^{-1}xH])\nu(\epsilon_{xH}) = \nu(\epsilon_{xuH})V([HaH])\nu(\epsilon_{xH})$ is a partial isometry with initial projection $\nu(\epsilon_{xH})$ and final projection $\nu(\epsilon_{xuH})$. These final projections are orthogonal for each of the $R(a)$ different cosets $uH \subseteq Ha^{-1}H$. For $h \in \nu(\epsilon_{xH})\mathcal{H}$, we have $h = \nu(\epsilon_{xH})h$, and hence

$$\begin{aligned}
\|V([HaH])h\|^2 &= \|V([HaH])\nu(\epsilon_{xH})h\|^2 \\
&= \Big\| \sum_{uH \subseteq Ha^{-1}H} \nu(\epsilon_{xuH})V([HaH])\nu(\epsilon_{xH})h \Big\|^2 \\
&= \sum_{uH \subseteq Ha^{-1}H} \|\nu(\epsilon_{xuH})V([HaH])\nu(\epsilon_{xH})h\|^2 \\
&= \sum_{uH \subseteq Ha^{-1}H} \|h\|^2 = R(a)\|h\|^2.
\end{aligned}$$

Thus (i) implies (iv) for matrix unit pairs.

Finally, suppose (ν, V) is a matrix unit pair such that (iv) holds. Fix $a, x \in G$ and set $P = \sum_{uH \subseteq Ha^{-1}H} \nu(\epsilon_{xuH})$; note that P is a (self-adjoint) projection in $B(\mathcal{H})$ because the $\nu(\epsilon_{xuH})$'s are mutually orthogonal projections. Then for $h \in \mathcal{H}$,

$$\begin{aligned}
R(a)\|\nu(\epsilon_{xH})h\|^2 &\geq \|V([HaH])|_{\nu(\epsilon_{xH})\mathcal{H}}\|^2 \|\nu(\epsilon_{xH})h\|^2 \\
&\geq \|V([HaH])\nu(\epsilon_{xH})h\|^2 \\
&= \|PV([HaH])\nu(\epsilon_{xH})h\|^2 + \|(1-P)V([HaH])\nu(\epsilon_{xH})h\|^2 \\
&= \Big(\sum_{uH \subseteq Ha^{-1}H} \|\nu(\epsilon_{xuH})V([HaH])\nu(\epsilon_{xH})h\|^2 \Big) + \|(1-P)V([HaH])\nu(\epsilon_{xH})h\|^2 \\
&= \Big(\sum_{uH \subseteq Ha^{-1}H} \|\nu(\epsilon_{xH})h\|^2 \Big) + \|(1-P)V([HaH])\nu(\epsilon_{xH})h\|^2
\end{aligned}$$

(since each matrix unit $\nu(\epsilon_{xuH})V([HaH])\nu(\epsilon_{xH})$ is a partial isometry whose initial space contains $\nu(\epsilon_{xH})h$)

$$= R(a)\|\nu(\epsilon_{xH})h\|^2 + \|(1-P)V([HaH])\nu(\epsilon_{xH})h\|^2.$$

This forces $(1-P)V([HaH])\nu(\epsilon_{xH})h = 0$. Since $h \in \mathcal{H}$ was arbitrary, this shows that $V([HaH])\nu(\epsilon_{xH}) = PV([HaH])\nu(\epsilon_{xH})$, which is precisely (i). This completes the proof. □

Proposition 1.4. *Let (G, H) be a discrete Hecke pair. The covariant pairs for (G, H) are precisely the matrix unit pairs which satisfy the equivalent conditions of Lemma* 1.3.

Proof. For brevity, we set $\upsilon_{xH,yH} = \nu(\epsilon_{xH})V([Hx^{-1}yH])\nu(\epsilon_{yH})$, so (ν, V) is a matrix unit pair if and only if $\{\upsilon_{xH,yH} \mid xH, yH \in G/H\}$ is a set of matrix units.

First suppose (ν, V) is a covariant pair. Then for $x, y, z \in G$ we have

$$\begin{aligned}\upsilon_{xH,yH}\upsilon_{yH,zH} &= \nu(\epsilon_{xH})V([Hx^{-1}yH])\nu(\epsilon_{yH})V([Hy^{-1}zH])\nu(\epsilon_{zH})\\ &= \nu(\epsilon_{xH})\Big(\sum_{\substack{uH\subseteq Hy^{-1}xH\\ vH\subseteq Hy^{-1}zH}} \nu(\epsilon_{yuH})V([Hu^{-1}vH])\nu(\epsilon_{yvH})\Big)\nu(\epsilon_{zH}).\end{aligned}$$

There is exactly one $uH \subseteq Hy^{-1}xH$ such that $xH = yuH$, namely $uH = y^{-1}xH$; similarly $vH = y^{-1}zH$ is the unique left coset in $Hy^{-1}zH$ such that $yvH = zH$. Thus the sums disappear, and $Hu^{-1}vH = Hx^{-1}yy^{-1}zH = Hx^{-1}zH$, so the above expression reduces to $\nu(\epsilon_{xH})V([Hx^{-1}zH])\nu(\epsilon_{zH}) = \upsilon_{xH,zH}$. Since also $\upsilon^*_{xH,yH} = \upsilon_{yH,xH}$ and $\upsilon_{xH,yH}\upsilon_{wH,zH} = 0$ unless $yH = wH$ (simply because ν and V are $*$-homomorphisms), this shows that (ν, V) is a matrix unit pair. Taking $b \in H$ in the covariant pair condition (1.1) shows that (ν, V) satisfies condition (i) of Lemma 1.3.

Conversely, suppose that (ν, V) is a matrix unit pair which satisfies the equivalent conditions of Lemma 1.3. Note that for any $a, x, b \in G$, and for any $uH \subseteq Ha^{-1}H$ and $vH \subseteq HbH$, we have $H(xu)^{-1}xH = HaH$ and $Hx^{-1}(xv)H = HbH$. Thus, conditions (i) and (ii) of Lemma 1.3 and the matrix unit assumption give

$$\begin{aligned}V([HaH])\nu(\epsilon_{xH})V([HbH]) &= V([HaH])\nu(\epsilon_{xH})\,\nu(\epsilon_{xH})V([HbH])\\ &= \sum_{uH\subseteq Ha^{-1}H} \nu(\epsilon_{xuH})V([HaH])\nu(\epsilon_{xH}) \sum_{vH\subseteq HbH} \nu(\epsilon_{xH})V([HbH])\nu(\epsilon_{xvH})\\ &= \sum_{\substack{uH\subseteq Ha^{-1}H\\ vH\subseteq HbH}} \nu(\epsilon_{xuH})V([H(xu)^{-1}xH])\nu(\epsilon_{xH})\,\nu(\epsilon_{xH})V([Hx^{-1}(xv)H])\nu(\epsilon_{xvH})\\ &= \sum_{\substack{uH\subseteq Ha^{-1}H\\ vH\subseteq HbH}} \upsilon_{xuH,xH}\,\upsilon_{xH,xvH} \quad = \sum_{\substack{uH\subseteq Ha^{-1}H\\ vH\subseteq HbH}} \upsilon_{xuH,xvH}\\ &= \sum_{\substack{uH\subseteq Ha^{-1}H\\ vH\subseteq HbH}} \nu(\epsilon_{xuH})V([Hu^{-1}vH])\nu(\epsilon_{xvH}).\end{aligned}$$

Hence (ν, V) is a covariant pair. □

Example 1.5. Let M be the representation of $c_0(G/H)$ on $\ell^2(G/H)$ by pointwise multiplication, and let ρ be the representation of $\mathcal{H}(G, H)$ on $\ell^2(G/H)$ by right convolution, so that

$$M(f)(\epsilon_{yH}) = f(yH)\epsilon_{yH} \quad \text{and} \quad \rho([HaH])(\epsilon_{yH}) = \sum_{uH\subseteq Ha^{-1}H} \epsilon_{yuH} = \chi_{yHa^{-1}H}$$

for $f \in c_0(G/H)$ and $a, y \in G$. (If H is normal in G, then ρ is the representation of the group algebra $\mathbb{C}(G/H)$ corresponding to the right regular representation of G/H.) Then (M, ρ) is a covariant pair. One way to see this is to first compute directly that for any $a, y \in G$,

$$\rho([HaH])M(\epsilon_{yH}) = \sum_{uH\subseteq Ha^{-1}H} \epsilon_{yuH} \otimes \overline{\epsilon_{yH}}, \tag{1.4}$$

where by definition $\xi \otimes \overline{\eta}(\zeta) = (\zeta \mid \eta)\xi$ for $\xi, \eta, \zeta \in \ell^2(G/H)$. Since $M(\epsilon_{xH}) = \epsilon_{xH} \otimes \overline{\epsilon_{xH}}$ for any $x \in G$, this gives

$$M(\epsilon_{xH})\rho([Hx^{-1}yH])M(\epsilon_{yH}) = \sum_{uH \subseteq Hy^{-1}xH} (\epsilon_{xH} \otimes \overline{\epsilon_{xH}})(\epsilon_{yuH} \otimes \overline{\epsilon_{yH}}) = \epsilon_{xH} \otimes \overline{\epsilon_{yH}}. \quad (1.5)$$

Thus (M, ρ) is a matrix unit pair which by (1.4) and (1.5) satisfies condition (i) of Lemma 1.3, and hence is a covariant pair by Proposition 1.4.

Theorem 1.6. *Let (G, H) be a discrete Hecke pair, let ν be a nondegenerate $*$-representation of $c_0(G/H)$, and let V be a unital $*$-representation of $\mathcal{H}(G, H)$ on the same Hilbert space $\mathcal{H}$. Then:*

(i) *(ν, V) is a matrix unit pair if and only if there exists a Hilbert space $\mathcal{H}_0$ and a representation $\tilde{V}$ of $\mathcal{H}(G, H)$ on $\mathcal{H}_0 \otimes \ell^2(G/H)$ such that (ν, V) is unitarily equivalent to $(1 \otimes M, \tilde{V})$ and such that*

$$\begin{aligned}(1 \otimes M(\epsilon_{xH}))\tilde{V}([Hx^{-1}yH])(1 \otimes M(\epsilon_{yH}))& \\ = 1 \otimes (M(\epsilon_{xH})\rho([Hx^{-1}yH])M(\epsilon_{yH}))& \quad \text{for all } x, y \in G.\end{aligned} \quad (1.6)$$

(ii) *(ν, V) is a covariant pair if and only if there exists a Hilbert space $\mathcal{H}_0$ such that (ν, V) is unitarily equivalent to the covariant representation $(1 \otimes M, 1 \otimes \rho)$ on $\mathcal{H}_0 \otimes \ell^2(G/H)$.*

In particular, a representation ν of $c_0(G/H)$ is equivalent to a multiple of M if and only if there exists a representation V of $\mathcal{H}(G, H)$ such that (ν, V) is a covariant pair.

Proof. Suppose first that (ν, V) is a matrix unit pair on $\mathcal{H}$, and write $\upsilon_{xH,yH}$ for each matrix unit $\nu(\epsilon_{xH})V([Hx^{-1}yH])\nu(\epsilon_{yH})$. Now set $\mathcal{H}_0 = \nu(\epsilon_H)\mathcal{H}$. Then it is straightforward to verify, using nondegeneracy of ν, that the rule

$$\nu(\epsilon_{zH})\xi \mapsto \upsilon_{H,zH}\xi \otimes \epsilon_{zH} \qquad (\xi \in \mathcal{H})$$

determines a unitary isomorphism Ψ of $\mathcal{H}$ onto $\mathcal{H}_0 \otimes \ell^2(G/H)$ such that

$$\operatorname{Ad}\Psi(\upsilon_{xH,yH}) = 1 \otimes (\epsilon_{xH} \otimes \overline{\epsilon_{yH}})$$

for all $x, y \in G$. (Equivalently, write $\mathcal{H} = \bigoplus_{zH \in G/H} \nu(\epsilon_{zH})\mathcal{H}$ and use $\bigoplus_{zH \in G/H} \upsilon_{H,zH}$ to map $\mathcal{H}$ onto $\bigoplus_{zH \in G/H} \nu(\epsilon_H)\mathcal{H} \cong \mathcal{H}_0 \otimes \ell^2(G/H)$.) In particular, for each $x \in G$, we have

$$\operatorname{Ad}\Psi(\nu(\epsilon_{xH})) = \operatorname{Ad}\Psi(\upsilon_{xH,xH}) = 1 \otimes (\epsilon_{xH} \otimes \overline{\epsilon_{xH}}) = 1 \otimes M(\epsilon_{xH});$$

thus $\operatorname{Ad}\Psi \circ \nu = 1 \otimes M$.

Now let $\tilde{V} = \operatorname{Ad}\Psi \circ V$ be the representation of $\mathcal{H}(G, H)$ on $\mathcal{H}_0 \otimes \ell^2(G/H)$ corresponding to V. Then for each $x, y \in G$,

$$\begin{aligned}(1 \otimes M(\epsilon_{xH}))\tilde{V}([Hx^{-1}yH])(1 \otimes M(\epsilon_{yH})) &= \operatorname{Ad}\Psi(\nu(\epsilon_{xH})V([Hx^{-1}yH])\nu(\epsilon_{yH})) \\ &= \operatorname{Ad}\Psi(\upsilon_{xH,yH}) \\ &= 1 \otimes (\epsilon_{xH} \otimes \overline{\epsilon_{yH}}) \\ &= 1 \otimes (M(\epsilon_{xH})\rho([Hx^{-1}yH])M(\epsilon_{yH})),\end{aligned}$$

using (1.5) for the last equality. This proves the forward implication in (i); the converse is straightforward because (1.6) implies that $(1 \otimes M, \tilde{V})$ is a matrix unit pair.

Now suppose (ν, V) is a covariant pair. Then in particular (ν, V) is a matrix unit pair, so we have $\mathcal{H}_0$, Ψ, and $\tilde{V}$ as in part (i). But now $(1\otimes M, \tilde{V}) = (\operatorname{Ad}\Psi\circ\nu, \operatorname{Ad}\Psi\circ V)$ is covariant because (ν, V) is, and it follows that

$$(1\otimes M(\epsilon_{xH}))\tilde{V}([HbH])(1\otimes M(\epsilon_{yH})) = (1\otimes M(\epsilon_{xH}))(1\otimes \rho([HbH]))(1\otimes M(\epsilon_{yH}))$$

for *every* $x, b, y \in G$: this is just (1.6) when $HbH = Hx^{-1}yH$, and both expressions are zero otherwise by Proposition 1.4. Letting x and y vary now shows that $\tilde{V}([HbH]) = 1\otimes\rho([HbH])$ for each b, so $\tilde{V} = 1\otimes\rho$. This proves the forward implication of (ii), and the converse is immediate because $(1\otimes M, 1\otimes\rho)$ is a covariant pair.

For the last statement of the theorem, it only remains to observe that if Ψ is a unitary operator intertwining ν and $1\otimes M$, then $V = \operatorname{Ad}\Psi^*\circ(1\otimes\rho)$ is a representation of $\mathcal{H}(G,H)$ such that (ν, V) is a covariant pair. □

Remark 1.7. Although it is not necessarily true that every matrix unit pair is a covariant pair, it follows easily from Theorem 1.6 that for for every matrix unit pair (ν, W), there exists a representation V of $\mathcal{H}(G,H)$ such that (ν, V) is a covariant pair.

2. Hecke algebras which are semigroup crossed products

Throughout this section, we consider a cancellative semigroup S satisfying the Ore condition $Ss\cap St \neq \emptyset$ for all $s, t\in S$, and we let $Q = S^{-1}S$; since S is cancellative, the Ore condition implies that Q is a group containing S as a sub-semigroup ([5, Theorem 1.24]). We suppose that there is an action of Q by automorphisms of a group N, and we let $G = N\rtimes Q$ be the semi-direct product group. Finally, we let H be a normal subgroup of N such that $sH = HsH$ for all $s\in S$ and such that there are finitely many right cosets of H in HsH for each $s\in S$. Thus (G, H) is a Hecke pair by [16, Proposition 1.7].[1] The Hecke algebra of Bost and Connes [2] comes from a Hecke pair of this sort, with $S = \mathbb{N}^*$, $Q = \mathbb{Q}^*_+$ acting on $N = \mathbb{Q}$ by $q\cdot n = n/q$, and $H = \mathbb{Z}$; see also [3, Example 4.3] and [16, Example 2.2].

In this situation, the Hecke algebra $\mathcal{H}(G,H)$ is isomorphic to a certain semigroup crossed product $\mathbb{C}(N/H)\times_\alpha S$, and thus each representation V of $\mathcal{H}(G,H)$ corresponds to a suitably covariant pair (U, W) of representations of N/H and S. The main result of this section (Theorem 2.1) shows that the covariance condition (1.1) for a pair (ν, V) can be expressed very naturally in terms of a set of more-familiar covariance conditions involving only ν, U, and W.

The precise statement of Theorem 2.1 requires a few preliminaries, which we have endeavored to keep brief; further discussion of their significance follows the proof of the theorem. First, since $s^{-1}Hs\subseteq H$ for $s\in S$, the formula $(xH)\cdot s = xsH$ defines a right action of S on G/H. We denote by rt the associated action of S by endomorphisms of $c_0(G/H)$, so that $\mathrm{rt}_s(f)(xH) = f(xsH)$ for $s\in S$ and $x\in G$. We also use rt to denote the action of the group N/H by automorphisms of $c_0(G/H)$ induced from the natural action of N/H on the right of G/H.

[1] To reconcile our assumption that $|H\backslash HsH| < \infty$ with the assumption that $|s^{-1}Hs\backslash H| < \infty$ in [16], note that for any subgroup H of any group G, and for any $x\in G$, the map $Hxh\mapsto x^{-1}Hxh$ is a bijection of $H\backslash HxH$ onto $x^{-1}Hx\backslash H$.

Next, suppose ν is a a nondegenerate representation of $c_0(G/H)$ and W is a representation of S by isometries on the same Hilbert space. Motivated by [20], we say that the pair (ν, W) is *Murphy-covariant* for $(c_0(G/H), S, \mathrm{rt})$ if

$$W_s\nu(f) = \nu(\mathrm{rt}_s(f))W_s \quad \text{for all } s \in S \text{ and } f \in c_0(G/H).$$

(This is a special case of (2.15) below.)

Lastly, for $s \in S$ and $n \in N$, define elements μ_s and $e(nH)$ of $\mathcal{H}(G, H)$ by

$$\mu_s = R(s)^{-1/2}[HsH] \quad \text{and} \quad e(nH) = [HnH],$$

where R is the right coset counting map. Then the map $\mu\colon S \to \mathcal{H}(G, H)$ is a representation of S by (algebraic) isometries, and $e\colon N/H \to \mathcal{H}(G, H)$ is a unitary representation of N/H ([16, Theorem 1.9]).

Theorem 2.1. *With S, N, G, and H as described above, let ν be a nondegenerate $*$-representation of $c_0(G/H)$, and let V be a unital $*$-representation of $\mathcal{H}(G, H)$ on the same space. Then (ν, V) is a covariant pair if and only if*

(i) *$(\nu, V \circ e)$ is covariant for $(c_0(G/H), N/H, \mathrm{rt})$, and*
(ii) *$(\nu, V \circ \mu)$ is Murphy-covariant for $(c_0(G/H), S, \mathrm{rt})$.*

Observe that (i) is equivalent to the condition

$$V(e(nH))\nu(\epsilon_{xH})V(e(nH))^* = \nu(\epsilon_{xn^{-1}H}) \quad \text{for all } n \in N \text{ and } x \in G. \tag{2.1}$$

Also, for $x, y \in G$ and $s \in S$ we have

$$xsH = yH \iff x \in yHs^{-1} = yHs^{-1}H \iff xH \subseteq yHs^{-1}H. \tag{2.2}$$

(so the endomorphism of G/H determined by s is precisely $R(s)$-to-one), and it follows that the endomorphism rt_s of $c_0(G/H)$ is given on characteristic functions by

$$\mathrm{rt}_s(\epsilon_{xH}) = \sum_{uH \subseteq Hs^{-1}H} \epsilon_{xuH}.$$

Thus condition (ii) of Theorem 2.1 is equivalent to

$$V([HsH])\nu(\epsilon_{xH}) = \sum_{uH \subseteq Hs^{-1}H} \nu(\epsilon_{xuH})V([HsH]) \quad \text{for all } s \in S \text{ and } x \in G, \tag{2.3}$$

and hence also (on taking adjoints) to

$$\nu(\epsilon_{xH})V([Hs^{-1}H]) = \sum_{uH \subseteq Hs^{-1}H} V([Hs^{-1}H])\nu(\epsilon_{xuH}) \quad \text{for all } s \in S \text{ and } x \in G. \tag{2.4}$$

The proof of Theorem 2.1 depends on the following lemma, which essentially shows that certain key operators behave like matrix units. First recall that for any discrete Hecke pair (G, H), the product in the Hecke algebra satisfies

$$[HaH][HbH](HxH) = \left|(HaH \cap xHb^{-1}H)/H\right| \tag{2.5}$$

for all $a, x, b \in G$; if it happens that $HaHbH = HabH$, then

$$[HaH][HbH] = \frac{R(a)R(b)}{R(ab)}[HabH] \tag{2.6}$$

by [17, Lemma 1] (see also [1, Corollary 3.3]). In our situation, for any $s,t\in S$ and $n,m\in N$, taking $a=s^{-1}n$ and $b=mt$ in the above gives the counting formula

$$\big|(Hs^{-1}nH\cap xHt^{-1}m^{-1}H)/H\big| = \frac{R(s^{-1}n)R(mt)}{R(s^{-1}nmt)}[Hs^{-1}nmtH](HxH) = \frac{R(t)}{R(s^{-1}t)} \tag{2.7}$$

for any $xH \subseteq Hs^{-1}nmtH$. (This last identity incorporates the observation that $R(s^{-1}n)=1$ for all $s\in S$ and $n\in N$, and more generally that $R(xny)=R(xy)$ for any $x,y\in G$ and $n\in N$. To see this, note that since N is normal in G, $xny=mxy$ for $m=xnx^{-1}\in N$, and since H is normal in N, the rule $Hxyh\mapsto Hmxyh$ gives a well-defined bijection of $H\backslash HxyH$ onto $H\backslash HmxyH = H\backslash HxnyH$.)

A direct calculation with (2.5) (as in the proof of [16, Theorem 1.9]) shows that for $s\in S$,

$$[HsH][Hs^{-1}H] = \chi_{sHs^{-1}} = \sum_{mH\subseteq sHs^{-1}H} e(mH). \tag{2.8}$$

Also, combining (2.6) and (2.7) yields the factorisation

$$[Hs^{-1}H]e(nH)[HtH] = \frac{R(t)}{R(s^{-1}t)}[Hs^{-1}ntH] \tag{2.9}$$

for $s,t\in S$ and $n\in N$. In the important special case where $nH\subseteq tHt^{-1}H$, so that $HntH=HtH$ and $Ht^{-1}nH = Ht^{-1}H$, we have

$$e(nH)[HtH]=[HtH] \qquad \text{and} \qquad [Ht^{-1}H]e(nH)=[Ht^{-1}H]. \tag{2.10}$$

This turns out to be the crux of the proof of Lemma 2.2.

Lemma 2.2. *If* (ν,V) *satisfies the hypotheses and conditions* (i) *and* (ii) *of Theorem* 2.1*, then*

$$V([Hs^{-1}H])\nu(\epsilon_{zH})V(e(kH))V([HrH]) = \nu(\epsilon_{zsH})V([Hs^{-1}krH])\nu(\epsilon_{zkrH})$$

for each $s,r\in S$*,* $k\in N$*, and* $z\in G$*.*

Proof. By (2.9), then (2.3) and (2.4), followed by (2.1),

$$\begin{aligned}
&\frac{R(r)}{R(s^{-1}r)}\nu(\epsilon_{zsH})V([Hs^{-1}krH])\nu(\epsilon_{zkrH}) \qquad (2.11)\\
&\qquad = \nu(\epsilon_{zsH})V([Hs^{-1}H])V(e(kH))V([HrH])\nu(\epsilon_{zkrH})\\
&\qquad = \sum_{\substack{uH\subseteq Hs^{-1}H\\ vH\subseteq Hr^{-1}H}} V([Hs^{-1}H])\nu(\epsilon_{zsuH})V(e(kH))\nu(\epsilon_{zkrvH})V([HrH])\\
&\qquad = \sum_{\substack{uH\subseteq Hs^{-1}H\\ vH\subseteq Hr^{-1}H}} V([Hs^{-1}H])\nu(\epsilon_{zsuH})\nu(\epsilon_{zkrvk^{-1}H})V(e(kH))V([HrH]),
\end{aligned}$$

which, since each term is zero unless $vH = r^{-1}k^{-1}sukH\subseteq Hr^{-1}H$, reduces to

$$= \sum_{\substack{uH\subseteq Hs^{-1}H\cap\\ s^{-1}krHr^{-1}k^{-1}H}} V([Hs^{-1}H])\nu(\epsilon_{zsuH})V(e(kH))V([HrH]). \tag{2.12}$$

Each uH in the sum at (2.12) satisfies $suH \subseteq sHs^{-1}H$ and $k^{-1}sukH \subseteq rHr^{-1}H$, so repeated use of (2.10) (together with (2.1)) gives

$$\begin{aligned}
&V([Hs^{-1}H])\nu(\epsilon_{zsuH})V(e(kH))V([HrH])\\
&\qquad = V([Hs^{-1}H])V(e(suH))\nu(\epsilon_{zsuH})V(e(kH))V([HrH])\\
&\qquad = V([Hs^{-1}H])\nu(\epsilon_{zH})V(e(suH))V(e(kH))V([HrH])\\
&\qquad = V([Hs^{-1}H])\nu(\epsilon_{zH})V(e(kH))V(e(k^{-1}sukH))V([HrH])\\
&\qquad = V([Hs^{-1}H])\nu(\epsilon_{zH})V(e(kH))V([HrH]).
\end{aligned}$$

Thus the terms in the sum at (2.12) are all identical; by the counting formula (2.7) there are precisely $R(r)/R(s^{-1}r)$ of them, so the sum at (2.12) reduces to

$$\frac{R(r)}{R(s^{-1}r)}V([Hs^{-1}H])\nu(\epsilon_{zH})V(e(kH))V([HrH]).$$

Comparing this with (2.11) completes the proof of the lemma. □

Proof of Theorem 2.1. First suppose that (ν, V) is a covariant pair. To establish (i) and (ii), it suffices to check (2.1) and (2.3), so fix $x \in G$, $n \in N$, and $s \in S$. For (2.1), the covariant pair condition (1.1) gives

$$\begin{aligned}
V(e(nH))\nu(\epsilon_{xH})V(e(nH))^* &= V([HnH])\nu(\epsilon_{xH})V([Hn^{-1}H])\\
&= \nu(\epsilon_{xn^{-1}H})V([Hnn^{-1}H])\nu(\epsilon_{xn^{-1}H})\\
&= \nu(\epsilon_{xn^{-1}H}).
\end{aligned}$$

For (2.3), first note that for each $uH \subseteq Hs^{-1}H$ we have $xusH = xH$ and $Hu^{-1}H = HsH$, so (1.1) gives

$$\nu(\epsilon_{xuH})V([HsH]) = \nu(\epsilon_{xuH})V([HsH])\nu(\epsilon_{xusH}) = \nu(\epsilon_{xuH})V([Hu^{-1}H])\nu(\epsilon_{xH}).$$

Thus, using (1.1) again, we have

$$\begin{aligned}
V([HsH])\nu(\epsilon_{xH}) &= \sum_{uH\subseteq Hs^{-1}H} \nu(\epsilon_{xuH})V([Hu^{-1}H])\nu(\epsilon_{xH})\\
&= \sum_{uH\subseteq Hs^{-1}H} \nu(\epsilon_{xuH})V([HsH]).
\end{aligned}$$

Conversely, suppose that (ν, V) satisfies covariance conditions (i) and (ii), and hence (2.1), (2.3), and (2.4). Fix $a, x, b \in G$ and then, using the Ore condition, choose $s, t, r \in S$ and $n, m \in N$ such that $a = s^{-1}nt$ and $b = t^{-1}mr$.[2] Then (2.1) and (2.8) together imply that for each $pH, qH \subseteq Ht^{-1}H$,

$$\begin{aligned}
\nu(\epsilon_{xpH})V([HtH][Ht^{-1}H])\nu(\epsilon_{xqH}) &= \sum_{mH\subseteq tHt^{-1}H} \nu(\epsilon_{xpH})V(e(mH))\nu(\epsilon_{xqH})\\
&= \sum_{mH\subseteq tHt^{-1}H} V(e(mH))\nu(\epsilon_{xpmH})\nu(\epsilon_{xqH})\\
&= V(e(p^{-1}qH))\nu(\epsilon_{xqH}),
\end{aligned}$$

[2]First write $a = s_a^{-1}n_at_a$ and $b = t_b^{-1}m_br_b$. Since $St_a \cap St_b \neq \emptyset$, there exist $\sigma_a, \sigma_b \in S$ such that $\sigma_at_a = \sigma_bt_b$. Now set $s = \sigma_as_a$, $n = \sigma_an_a\sigma_a^{-1}$, $t = \sigma_at_a = \sigma_bt_b$, $m = \sigma_bm_b\sigma_b^{-1}$, and $r = \sigma_br_b$.

since $mH = p^{-1}qH$ is the unique left coset in $tHt^{-1}H$ such that $xpmH = xqH$. Using this with (2.3) and (2.4) gives

$$\begin{aligned} V([HtH])\nu(\epsilon_{xH})V([Ht^{-1}H]) &= \sum_{pH,qH\subseteq Ht^{-1}H} \nu(\epsilon_{xpH})V([HtH][Ht^{-1}H])\nu(\epsilon_{xqH}) \\ &= \sum_{pH,qH\subseteq Ht^{-1}H} V(e(p^{-1}qH))\nu(\epsilon_{xqH}). \end{aligned}$$

Thus, by the factorisation (2.9) and (2.1) again, we have

$$\begin{aligned} &\frac{R(t)}{R(s^{-1}t)}\frac{R(r)}{R(t^{-1}r)}V([HaH])\nu(\epsilon_{xH})V([HbH]) \qquad (2.13) \\ &\quad= \frac{R(t)}{R(s^{-1}t)}\frac{R(r)}{R(t^{-1}r)}V([Hs^{-1}ntH])\nu(\epsilon_{xH})V([Ht^{-1}mrH]) \\ &\quad= V([Hs^{-1}H])V(e(nH))V([HtH])\nu(\epsilon_{xH})V([Ht^{-1}H])V(e(mH))V([HrH]) \\ &\quad= \sum_{pH,qH\subseteq Ht^{-1}H} V([Hs^{-1}H])V(e(nH))V(e(p^{-1}qH))\nu(\epsilon_{xqH})V(e(mH))V([HrH]) \\ &\quad= \sum_{pH,qH\subseteq Ht^{-1}H} V([Hs^{-1}H])\nu(\epsilon_{xpn^{-1}H})V(e(np^{-1}qmH))V([HrH]), \end{aligned}$$

which, by Lemma 2.2, becomes

$$= \sum_{pH,qH\subseteq Ht^{-1}H} \nu(\epsilon_{xpn^{-1}sH})V([Hs^{-1}np^{-1}qmrH])\nu(\epsilon_{xqmrH}). \qquad (2.14)$$

The terms in this sum depend only on $pn^{-1}sH$, which is always contained in $Ht^{-1}n^{-1}sH = Ha^{-1}H$, and $qmrH$, which is always contained in $Ht^{-1}mrH = HbH$. Now for each $uH \subseteq Ha^{-1}H$, by (2.2) we have $pn^{-1}sH = uH$ if and only if $pH \subseteq uHs^{-1}nH$, so by the counting formula (2.7) the number of left cosets $pH \subseteq Ht^{-1}H$ such that $pn^{-1}sH = uH$ is

$$\big|(Ht^{-1}H \cap uHs^{-1}nH)/H\big| = \big|(Hs^{-1}nH \cap u^{-1}Ht^{-1}H)/H\big| = \frac{R(t)}{R(s^{-1}t)}.$$

Similarly, for each $vH \subseteq HbH$, the number of cosets $qH \subseteq Ht^{-1}H$ such that $qmrH = vH$ is $|(Ht^{-1}H \cap vHr^{-1}m^{-1}H)/H| = R(r)/R(t^{-1}r)$. Thus we can collect like terms and rewrite (2.14) as

$$\frac{R(t)}{R(s^{-1}t)}\frac{R(r)}{R(t^{-1}r)} \sum_{\substack{uH\subseteq Ha^{-1}H \\ vH\subseteq HbH}} \nu(\epsilon_{xuH})V([Hu^{-1}vH])\nu(\epsilon_{xvH});$$

a comparison with (2.13) completes the proof of Theorem 2.1. □

Murphy and Stacey covariance. If α is an action of a semigroup S on a C^*-algebra A, π is a nondegenerate representation A on a Hilbert space $\mathcal{H}$, and W is a representation of S by isometries of $\mathcal{H}$, we say that the pair (π, W) *Murphy-covariant* for (A, S, α) if

$$W_s\pi(a) = \pi(\alpha_s(a))W_s \quad \text{for all } s \in S \text{ and } a \in A. \tag{2.15}$$

We say that (π, W) is *Stacey-covariant* (after [23]) if

$$W_s\pi(a)W_s^* = \pi(\alpha_s(a)) \quad \text{for all } s \in S \text{ and } a \in A. \tag{2.16}$$

Clearly Stacey-covariance implies Murphy-covariance, but the converse is not true in general. For instance, it follows from Example 1.5 and Theorem 2.1 that, in the context of that theorem, $(M, \rho \circ \mu)$ is Murphy-covariant for $(c_0(G/H), S, \mathrm{rt})$. Since the endomorphisms rt_s extend to unital endomorphisms of $c_b(G/H) = M(c_0(G/H))$, Stacey-covariance would imply that each $\rho(\mu_s)$ is in fact unitary. However, $\rho(\mu_s)$ cannot be unitary unless $R(s) = 1$, since by (2.8), for each $\epsilon_{xH} \in \ell^2(G/H)$ we have

$$\begin{aligned}
\rho(\mu_s)\rho(\mu_s)^*(\epsilon_{xH}) &= \frac{1}{R(s)}\rho([HsH][Hs^{-1}H])(\epsilon_{xH}) \\
&= \frac{1}{R(s)} \sum_{mH \subseteq sHs^{-1}H} \rho(e(mH))(\epsilon_{xH}) \\
&= \frac{1}{R(s)} \sum_{mH \subseteq sHs^{-1}H} \epsilon_{xm^{-1}H} = \frac{1}{R(s)}\chi_{xsHs^{-1}H}.
\end{aligned}$$

Exel covariance. In recent work of Exel ([10]), a crossed product $A \times_{\alpha,L} \mathbb{N}$ is associated to each pair (α, L) consisting of an endomorphism α of a C^*-algebra A and a "transfer function" L for α. In [4] (and implicitly in [10]), it was shown that Exel's crossed product is universal for a family of representations which are, among other things, Murphy-covariant for the semigroup action $(A, \mathbb{N}, \alpha)$. As we will show below, this family includes representations which arise naturally from covariant pairs.

In the context of Theorem 2.1, fix $s \in S$ and define $L_s\colon c_0(G/H) \to c_0(G/H)$ by

$$L_s(f)(xH) = \frac{1}{R(s)} \sum_{yH \in G/H\colon ysH = xH} f(yH). \tag{2.17}$$

Then L_s is a positive bounded linear map, and it is a *transfer function* for the endomorphism rt_s in the sense that $L_s(\mathrm{rt}_s(g)f) = gL_s(f)$ for $f, g \in c_0(G/H)$, since for $xH \in G/H$ we have

$$L_s(\mathrm{rt}_s(g)f)(xH) = \frac{1}{R(s)} \sum_{yH \in G/H\colon ysH = xH} g(ysH)f(yH) = g(xH)L_s(f)(xH).$$

In following proof, we will frequently make use of the easily-verified formula

$$L_s(\epsilon_{zH}) = \frac{1}{R(s)}\epsilon_{zsH}. \tag{2.18}$$

Proposition 2.3. *Let (ν, V) be a covariant pair for (G, H), with G, H, and S as in Theorem* 2.1. *Then for each $s \in S$, the pair $(\nu, V(\mu_s))$ is a covariant representation of $(c_0(G/H), \mathrm{rt}_s, L_s)$ in the sense of* [4, Definition 3.8].

Proof. We need to verify conditions (TC1) and (TC2) of [4, Definition 3.1], and condition (C3) of [4, Definition 3.8]. The first of these follows immediately from the Murphy-covariance of $(\nu, V\circ\mu)$ in Theorem 2.1. For each $zH \in G/H$, taking $r = s$ and $k = e$ in Lemma 2.2 and invoking (2.18) gives

$$\begin{aligned}V(\mu_s)^*\nu(\epsilon_{zH})V(\mu_s) &= \frac{1}{R(s)}V([Hs^{-1}H])\nu(\epsilon_{zH})V([HsH])\\ &= \frac{1}{R(s)}\nu(\epsilon_{zsH}) = \nu(L_s(\epsilon_{zH})),\end{aligned}$$

which implies (TC2). (Thus $(\nu, V(\mu_s))$ is a *Toeplitz-covariant* representation of $(c_0(G/H), \mathrm{rt}_s, L_s)$).

To verify the remaining covariance condition (C3), we need to know which elements $f \in c_0(G/H)$ act as compact operators on the right-Hilbert $c_0(G/H)$-module M_{L_s} obtained by completing $c_0(G/H)$ in the norm defined by the $c_0(G/H)$-valued inner product $\langle f, g\rangle_{L_s} = L_s(f^*g)$ (see the start of [4, §3]). Write ϕ_s for the homomorphism of $c_0(G/H)$ into $\mathcal{L}(M_{L_s})$ defined by the left action of $c_0(G/H)$, which is given by ordinary multiplication on $c_0(G/H) \subseteq M_{L_s}$. We claim that $\phi_s(\epsilon_{zH})$ is the rank-one operator $R(s)\Theta_{\epsilon_{zH},\epsilon_{zH}}$, where by definition $\Theta_{g,h}(f) = g\langle h, f\rangle_{L_s}$. Indeed, for $f \in c_0(G/H) \subseteq M_{L_s}$ and $xH \in G/H$ we have, using (2.18) again,

$$\begin{aligned}R(s)\Theta_{\epsilon_{zH},\epsilon_{zH}}(f)(xH) &= R(s)(\epsilon_{zH}\langle\epsilon_{zH}, f\rangle)_{L_s}(xH)\\ &= R(s)\epsilon_{zH}(xH)\mathrm{rt}_s(L_s(\epsilon_{zH}^*f))(xH)\\ &= R(s)\epsilon_{zH}(xH)\mathrm{rt}_s(f(zH)L_s(\epsilon_{zH}))(xH)\\ &= R(s)\epsilon_{zH}(xH)f(zH)L_s(\epsilon_{zH})(xsH)\\ &= \epsilon_{zH}(xH)f(zH)\epsilon_{zsH}(xsH)\\ &= \epsilon_{zH}(xH)f(xH),\end{aligned}$$

which is just $(\phi_s(\epsilon_{zH})f)(xH)$, proving the claim. It follows that ϕ_s takes values in $\mathcal{K}(M_{L_s})$, so the ideal K_{rt_s} appearing in [4, Definition 3.8] is all of $c_0(G/H)$.

The covariance condition (C3) is expressed in terms of the representation ψ_s of M_{L_s} defined on the dense subspace $c_0(G/H)$ by $\psi_s(f) = \nu(f)V(\mu_s)$. ($\psi_s$ would be denoted by $\psi_{V(\mu_s)}$ in [4].) It requires that

$$(\psi_s, \nu)^{(1)}(\phi_s(f)) = \nu(f) \quad \text{for } f \in K_{\mathrm{rt}_s}, \tag{2.19}$$

where $(\psi_s, \nu)^{(1)}$ is the representation of $\mathcal{K}(M_{L_s})$ induced by the Toeplitz representation (ψ_s, ν) of M_{L_s} (see [4, §2 and Lemma 3.2]). By continuity it suffices to check (2.19) for $f = \epsilon_{zH}$. Using (2.8), we have

$$\begin{aligned}(\psi_s, \nu)^{(1)}(\phi_s(\epsilon_{zH})) &= R(s)(\psi_s, \nu)^{(1)}(\Theta_{\epsilon_{zH},\epsilon_{zH}})\\ &= R(s)\psi_s(\epsilon_{zH})\psi_s(\epsilon_{zH})^*\\ &= R(s)\nu(\epsilon_{zH})V(\mu_s)V(\mu_s)^*\nu(\epsilon_{zH})\\ &= \nu(\epsilon_{zH})V([HsH][Hs^{-1}H])\nu(\epsilon_{zH})\\ &= \sum_{mH\subseteq sHs^{-1}H}\nu(\epsilon_{zH})V(e(mH))\nu(\epsilon_{zH}),\end{aligned}$$

but by covariance property (i) in Theorem 2.1,

$$\nu(\epsilon_{zH})V(e(mH))\nu(\epsilon_{zH}) = V(e(mH))\nu(\epsilon_{zmH})\nu(\epsilon_{zH}) = \begin{cases} \nu(\epsilon_{zH}) & \text{if } mH = H \\ 0 & \text{otherwise.} \end{cases}$$

Thus the above sum collapses to the single term $\nu(\epsilon_{zH})$, as required. □

Remark 2.4. One can verify directly that $R(s)R(t) = R(ts)$ for $s, t \in S$. It then follows from another application of (2.18) that $L_s L_t = L_{ts}$, so (α, L) is an action of S on $c_0(G/H)$ in the sense of Larsen [19]. (The extra condition on extendibility of L_s required in [19] holds because the formula (2.17) also defines an operator L_s on $c_b(G/H) = M(c_0(G/H))$.)

3. The imprimitivity theorem

Following Echterhoff and Quigg in [9], let $\delta\colon B \to B \otimes C^*(G)$ be a coaction of a discrete group G on a C^*-algebra B, and let H be a subgroup of G. The spectral subspaces $B_x = \{b \in B \mid \delta(b) = b \otimes x\}$ form a Fell bundle $\mathcal{B}$ over G, and the direct product $\mathcal{B} \times G/H$ is a Fell bundle over the transformation groupoid $G \times G/H$. The $*$-algebra $\Gamma_c(\mathcal{B} \times G/H)$ of finitely supported sections of this bundle, which we identify with $\operatorname{span}\{(b, xH) \mid b \in B, xH \in G/H\}$, has an (universal) enveloping C^*-algebra which is denoted by $C^*(\mathcal{B} \times G/H)$ in [9].

Recall from [9, Definition 2.4] that representations π of B and ν of $c_0(G/H)$ on the same Hilbert space form a *covariant representation of* $(B, G/H, \delta|)$ if

$$\pi(b_x)\nu(\epsilon_{yH}) = \nu(\epsilon_{xyH})\pi(b_x) \quad \text{for all } x, y \in G, b_x \in B_x. \tag{3.1}$$

(Note that when H is not normal in G, the notation $\delta|$ is purely formal: it does not stand for an actual coaction.) For every covariant representation (π, ν) of $(B, G/H, \delta|)$ there is a unique representation $\pi \times \nu$ of $C^*(\mathcal{B} \times G/H)$, called the *integrated form* of (π, ν), such that $\pi \times \nu(b, xH) = \pi(b)\nu(\epsilon_{xH})$ for all $b \in B$ and $x \in G$. We shall assume that δ is maximal in the sense of [6]; since G is discrete this is equivalent to asking that the C^*-algebra B be isomorphic to $C^*(\mathcal{B})$, the enveloping C^*-algebra of $\Gamma_c(\mathcal{B})$ ([6, Proposition 4.2]). In this case, we know from [9, Proposition 2.7] that every representation of $C^*(\mathcal{B} \times G/H)$ comes from a covariant pair. This universal property implies in particular that when N is a normal subgroup of G — so that the restriction $\delta|$ is a coaction of the group G/N — $C^*(\mathcal{B} \times G/N)$ is isomorphic to the usual crossed product $B \times_{\delta|} (G/N)$ ([9, Corollary 2.8]). Thus for general H we feel free to write $B \times_{\delta|} (G/H)$ for $C^*(\mathcal{B} \times G/H)$, which we call the *Echterhoff-Quigg crossed product.*

Example 3.1. Let θ be a representation of B on a Hilbert space $\mathcal{H}_\theta$, and denote by λ the quasi-regular representation of G on $\ell^2(G/H)$. Then the pair $((\theta \otimes \lambda) \circ \delta, 1 \otimes M)$ is a covariant representation of $(B, G/H, \delta|)$ on $\mathcal{H}_\theta \otimes \ell^2(G/H)$. For $x, y \in G$, $\lambda_x(\epsilon_{yH}) = \epsilon_{xyH}$, so for $b_x \in B_x$ we have

$$\begin{aligned}(\theta \otimes \lambda) \circ \delta(b_x)(1 \otimes M)(\epsilon_{yH}) &= (\theta(b_x) \otimes \lambda_x)(1 \otimes M(\epsilon_{yH})) \\ &= \theta(b_x) \otimes (\lambda_x M(\epsilon_{yH})\lambda_x^* \lambda_x) \\ &= (1 \otimes M(\epsilon_{xyH}))(\theta(b_x) \otimes \lambda_x) \\ &= (1 \otimes M)(\epsilon_{xyH})(\theta \otimes \lambda) \circ \delta(b_x).\end{aligned}$$

Just as for normal subgroups, we call the integrated form $((\theta \otimes \lambda) \circ \delta) \times (1 \otimes M)$ of the covariant representation in Example 3.1 the *regular representation of* $B \times_{\delta|} (G/H)$ *induced by* θ. The following imprimitivity theorem characterises these representations for Hecke subgroups.

Theorem 3.2. *Let δ be a maximal coaction of a discrete group G on a C^*-algebra B, and let H be a Hecke subgroup of G. Suppose $\pi \times \nu$ is a representation of the Echterhoff-Quigg crossed product $B \times_{\delta|} (G/H)$ on a Hilbert space $\mathcal{H}$. Then $\pi \times \nu$ is equivalent to a regular representation if and only if there exists a representation V of $\mathcal{H}(G, H)$ on $\mathcal{H}$ such that (ν, V) is a covariant pair and such that the range of V commutes with the range of π.*

Proof. First suppose that $\pi \times \nu$ is equivalent to a regular representation, so there exists a representation $\theta \colon B \to B(\mathcal{H}_\theta)$ and a unitary isomorphism $\Psi \colon \mathcal{H} \to \mathcal{H}_\theta \otimes \ell^2(G/H)$ which intertwines $\pi \times \nu$ and $((\theta \otimes \lambda) \circ \delta) \times (1 \otimes M)$. Set $V = \operatorname{Ad} \Psi^* \circ (1 \otimes \rho)$; then (ν, V) is a covariant pair because it is equivalent to $(1 \otimes M, 1 \otimes \rho)$, and the ranges of π and V commute because the ranges of $(\theta \otimes \lambda) \circ \delta$ and $1 \otimes \rho$ commute. This proves the forward implication.

For the converse, suppose there exists V such that (ν, V) is a covariant pair and such that the ranges of π and V commute. Then by Theorem 1.6, there exists a Hilbert space $\mathcal{H}_0$ and a unitary isomorphism $\Psi \colon \mathcal{H} \to \mathcal{H}_0 \otimes \ell^2(G/H)$ which intertwines (ν, V) and $(1 \otimes M, 1 \otimes \rho)$. Set $\tilde{\pi} = \operatorname{Ad} \Psi \circ \pi$, and for conciseness, also set $\tilde{\nu} = \operatorname{Ad} \Psi \circ \nu = 1 \otimes M$ and $\tilde{V} = \operatorname{Ad} \Psi \circ V = 1 \otimes \rho$.

We need to produce a representation $\theta \colon B \to B(\mathcal{H}_0)$ such that $(\theta \otimes \lambda) \circ \delta = \tilde{\pi}$. To do this, first fix $z \in G$ and $b_z \in B_z$. We will show that $(1 \otimes \lambda_z^*)\tilde{\pi}(b_z)$ commutes with $1 \otimes \mathcal{K}(\ell^2(G/H))$; it will then follow that there exists $\theta(b_z) \in B(\mathcal{H}_0)$ such that $\theta(b_z) \otimes 1 = (1 \otimes \lambda_z^*)\tilde{\pi}(b_z)$. It suffices to check that $(1 \otimes \lambda_z^*)\tilde{\pi}(b_z)$ commutes with the operators (recall (1.5))

$$1 \otimes (\epsilon_{xH} \otimes \overline{\epsilon_{yH}}) = 1 \otimes (M(\epsilon_{xH})\rho([Hx^{-1}yH])M(\epsilon_{yH})) = \tilde{\nu}(\epsilon_{xH})\tilde{V}([Hx^{-1}yH])\tilde{\nu}(\epsilon_{yH})$$

for $x, y \in G$. Since $(\tilde{\pi}, \tilde{\nu})$ is covariant for $(B, G/H, \delta|)$, and since the ranges of $\tilde{\pi}$ and $\tilde{V}$ commute, we have

$$\begin{aligned}
(1 \otimes \lambda_z^*)\tilde{\pi}(b_z)(1 \otimes (\epsilon_{xH} \otimes \overline{\epsilon_{yH}})) &= (1 \otimes \lambda_z^*)\tilde{\pi}(b_z)\tilde{\nu}(\epsilon_{xH})\tilde{V}([Hx^{-1}yH])\tilde{\nu}(\epsilon_{yH}) \\
&= (1 \otimes \lambda_z^*)\tilde{\nu}(\epsilon_{zxH})\tilde{\pi}(b_z)\tilde{V}([Hx^{-1}yH])\tilde{\nu}(\epsilon_{yH}) \\
&= (1 \otimes \lambda_z^*)\tilde{\nu}(\epsilon_{zxH})\tilde{V}([Hx^{-1}yH])\tilde{\pi}(b_z)\tilde{\nu}(\epsilon_{yH}) \\
&= (1 \otimes \lambda_z^*)\tilde{\nu}(\epsilon_{zxH})\tilde{V}([Hx^{-1}yH])\tilde{\nu}(\epsilon_{zyH})\tilde{\pi}(b_z).
\end{aligned}$$

By the covariance of $(\tilde{\nu}, 1 \otimes \lambda) = (1 \otimes M, 1 \otimes \lambda)$ for the action of G by left translation on $c_0(G/H)$, and because the ranges of $\tilde{V} = 1 \otimes \rho$ and $1 \otimes \lambda$ commute, this is

$$\begin{aligned}
&= \tilde{\nu}(\epsilon_{xH})(1 \otimes \lambda_z^*)\tilde{V}([Hx^{-1}yH])\tilde{\nu}(\epsilon_{zyH})\tilde{\pi}(b_z) \\
&= \tilde{\nu}(\epsilon_{xH})\tilde{V}([Hx^{-1}yH])(1 \otimes \lambda_z^*)\tilde{\nu}(\epsilon_{zyH})\tilde{\pi}(b_z) \\
&= \tilde{\nu}(\epsilon_{xH})\tilde{V}([Hx^{-1}yH])\tilde{\nu}(\epsilon_{yH})(1 \otimes \lambda_z^*)\tilde{\pi}(b_z) \\
&= (1 \otimes (\epsilon_{xH} \otimes \overline{\epsilon_{yH}}))(1 \otimes \lambda_z^*)\tilde{\pi}(b_z).
\end{aligned}$$

Now linearly extend the assignment $b_z \mapsto \theta(b_z)$ to get a map θ on $\Gamma_c(\mathcal{B}) = \operatorname{span}\{b_z \mid z \in G, b_z \in B_z\}$ which, by construction, satisfies $(\theta \otimes \lambda) \circ \delta(b) = \tilde{\pi}(b)$ for $b \in \Gamma_c(\mathcal{B})$. Then θ is in fact a $*$-homomorphism, since for $b_z \in B_z$ (so that $(b_z)^* \in B_{z^{-1}}$) we have

$$\begin{aligned}\theta((b_z)^*) \otimes 1 &= (1 \otimes \lambda_{z^{-1}}^*)\tilde{\pi}((b_z)^*) = \tilde{\pi}((b_z)^*)(1 \otimes \lambda_{z^{-1}}^*)\\ &= \tilde{\pi}(b_z)^*(1 \otimes \lambda_z) = ((1 \otimes \lambda_z^*)\tilde{\pi}(b_z))^* = \theta(b_z)^* \otimes 1,\end{aligned}$$

and in addition, for $w \in G$ and $c_w \in B_w$ we have

$$\begin{aligned}\theta(b_z)\theta(c_w) \otimes 1 &= (\theta(b_z) \otimes 1)(1 \otimes \lambda_w^*)\tilde{\pi}(c_w) = (1 \otimes \lambda_w^*)(\theta(b_z) \otimes 1)\tilde{\pi}(c_w)\\ &= (1 \otimes \lambda_w^*)(1 \otimes \lambda_z^*)\tilde{\pi}(b_z)\tilde{\pi}(c_w) = (1 \otimes \lambda_{zw}^*)\tilde{\pi}(b_z c_w) = \theta(b_z c_w) \otimes 1.\end{aligned}$$

By the universal property of $C^*(\mathcal{B})$, θ therefore extends to a representation of $B \cong C^*(\mathcal{B})$ such that $(\theta \otimes \lambda) \circ \delta = \tilde{\pi}$, as desired. □

Suppose, as in Section 2, that G is a semidirect product $N \rtimes Q$, where $Q = S^{-1}S$ for an Ore semigroup S, and that H is a normal subgroup of N such that $sH = HsH$ and $|H\backslash HsH| < \infty$ for all $s \in S$. Then Theorem 3.2 can be re-formulated without reference to the Hecke algebra. To do so, we first recall from Theorem 1.9 and Corollary 1.12 of [16] (see also [17]) that there is an action α of S by injective corner endomorphisms of $\mathbb{C}(N/H)$ such that the pair (e, μ) appearing in Theorem 2.1 gives rise to an isomorphism $e \times \mu$ of the $*$-algebraic semigroup crossed product $\mathbb{C}(N/H) \times_\alpha S$ onto $\mathcal{H}(G, H)$.

Now, blurring the distinction between representations of N/H, $\mathbb{C}(N/H)$, and $C^*(N/H)$, it follows from the universal property of the crossed product that the map $V \mapsto (V \circ e, V \circ \mu)$ is a bijection between the set of unital $*$-representations of $\mathcal{H}(G, H)$ and the set of Stacey-covariant pairs (see (2.16)) for $(C^*(N/H), S, \alpha)$. Moreover, the range of a given representation V of $\mathcal{H}(G, H)$ is precisely the $*$-algebra generated by $V(e(N/H))$ and $V(\mu(S))$. The following corollary is now immediate from Theorem 3.2.

Corollary 3.3. *Let S, N, G, and H be as above, let δ be a maximal coaction of G on a C^*-algebra B, and let $\pi \times \nu$ be a representation of the Echterhoff-Quigg crossed product $B \times_{\delta|} (G/H)$ on a Hilbert space $\mathcal{H}$. Then $\pi \times \nu$ is equivalent to a regular representation if and only if there is a unitary representation $U\colon N/H \to B(\mathcal{H})$ and an isometric representation $W : S \to B(\mathcal{H})$ such that*

(i) *(U, W) is a Stacey-covariant representation of $(C^*(N/H), S, \alpha)$,*
(ii) *$U(N/H)$ and $W(S)$ commute with the range of π,*
(iii) *(ν, W) is a Murphy-covariant representation of $(c_0(G/H), S, \mathrm{rt})$, and*
(iv) *(ν, U) is a covariant representation of $(c_0(G/H), N/H, \mathrm{rt})$.*

It is interesting to see here a natural juxtaposition of Murphy-covariance and Stacey-covariance for the same semigroup.

4. Application to the extension of unitary representations

To apply our imprimitivity theorem to the extension problem considered in [12], we want to take (B, δ) to be a dual coaction $(A \times_\alpha G, \widehat{\alpha})$. We know from [6, Proposition 3.4] that $\widehat{\alpha}$ is maximal, so Theorem 3.2 applies, but we also need to know that the Echterhoff-Quigg crossed product $(A \times_\alpha G) \times_{\widehat{\alpha}|} (G/H)$ is isomorphic to the imprimitivity algebra $(A \otimes c_0(G/H)) \times_{\alpha \otimes \mathrm{lt}} G$ which appears in [12, Theorem 1]. (Throughout this section, lt denotes the action of G on $c_0(G/H)$ by left translation.) Related, but different, versions of the following proposition can be found in [8, Lemma 2.4] and [7, Theorem A.64] (for normal subgroups), and [8, Proposition 2.8] (for reduced crossed products).

Proposition 4.1. *Suppose α is an action of a discrete group G on a C^*-algebra A, and H is a subgroup of G. Then there is an isomorphism of the Echterhoff-Quigg crossed product $(A\times_\alpha G)\times_{\widehat{\alpha}|}(G/H)$ onto $(A\otimes c_0(G/H))\times_{\alpha\otimes\mathrm{lt}}G$ which carries each representation*

$$(\psi \times W) \times \nu \quad \textit{into} \quad (\psi \otimes \nu) \times W; \tag{4.1}$$

in particular, for each representation $\phi\times U$ of $A\times_\alpha G$ it carries the regular representation

$$(((\phi \times U) \otimes \lambda) \circ \widehat{\alpha}) \times (1 \otimes M) \quad \textit{into} \quad (\phi \otimes M) \times (U \otimes \lambda). \tag{4.2}$$

Remark 4.2. Equations (4.1) and (4.2) involve tensor product symbols with at least three different meanings. While in most cases the meaning is clear from context, there is potential for confusion when trying to compare the maximal tensor product $\psi \otimes \nu$ in (4.1) with the spatial tensor product $\phi \otimes M$ in (4.2); this can be resolved by writing $\phi \otimes M$ as $(\phi \otimes 1) \otimes_{\max} (1 \otimes M)$.

Proof of Proposition 4.1. For $x \in G$, the spectral subspace B_x in the crossed product $B = A\times_\alpha G$ is given in terms of the universal covariant representation $(i_A, i_G)\colon (A, G) \to M(A \times_\alpha G)$ by $B_x := \{i_A(a)i_G(x) \mid a \in A\}$. Thus (using (3.1)) a pair $(\psi \times W, \nu)$ is a covariant representation of $(A \times_\alpha G, G/H, \widehat{\alpha}|)$ if and only if

$$\psi(a)W_x\nu(\epsilon_{yH}) = \nu(\epsilon_{xyH})\psi(a)W_x \quad \text{for all } x, y \in G \text{ and } a \in A,$$

which happens if and only if the ranges of ψ and ν commute elementwise and (ν, W) is a covariant representation of $(c_0(G/H), G, \mathrm{lt})$. In particular, this observation implies that the canonical embeddings $(k_A, k_G, k_{c(G/H)})$ of $(A, G, c_0(G/H))$ in $M((A\otimes c_0(G/H))\times_{\alpha\otimes\mathrm{lt}} G)$ generate a covariant representation $(k_A \times k_G, k_{c(G/H)})$ of $(A \times_\alpha G, G/H, \widehat{\alpha}|)$, and hence by [9, Proposition 2.7] induce a homomorphism Λ of $(A \times_\alpha G) \times_{\widehat{\alpha}|} (G/H)$ into $M((A \otimes c_0(G/H)) \times_{\alpha\otimes\mathrm{lt}} G)$ such that

$$\Lambda(i_A(a)i_G(x), yH) = k_A(a)k_G(x)k_{c(G/H)}(\epsilon_{yH}). \tag{4.3}$$

We shall prove that Λ is the required isomorphism.

First observe that elements of the form (4.3) belong to and span a dense subspace of $(A\otimes c_0(G/H)) \times_{\alpha\otimes\mathrm{lt}} G$, so Λ is surjective. To see that Λ is injective, we show that every representation of $(A \times_\alpha G) \times_{\widehat{\alpha}} (G/H)$ factors through Λ. Since $\widehat{\alpha}$ is maximal, we know from [9, Proposition 2.7] that every such representation has the form $\theta \times \nu$ for some covariant pair (θ, ν), and we can write θ as $\psi \times W$ for some covariant representation (ψ, W) of (A, G, α). The description of the covariant representations in the previous paragraph implies, first, that ψ and ν combine to give a representation $\psi \otimes \nu$ of $A \otimes c_0(G/H)$ (since their ranges commute), and second, that $(\psi \otimes \nu, W)$ is covariant for $\alpha \otimes \mathrm{lt}$ (since (ψ, W) is covariant for α and (ν, W) is covariant for lt). Now we just need to check using (4.3) that

$$((\psi \otimes \nu) \times W) \circ \Lambda = (\psi \times W) \times \nu, \tag{4.4}$$

and deduce that Λ is injective. The formula (4.4) immediately gives (4.1).

To see what Λ does to the regular representation induced by a representation $\phi \times U$ of $A \times_\alpha G$, recall that the dual coaction $\widehat{\alpha}$ satisfies $\widehat{\alpha}(i_A(a)) = i_A(a) \otimes 1$ and $\widehat{\alpha}(i_G(x)) =$

$i_G(x) \otimes u(x)$, where $u\colon G \to C^*(G)$ is the canonical map. So $((\phi \times U) \otimes \lambda) \circ \widehat{\alpha} = (\phi \otimes 1) \times (U \otimes \lambda)$, and thus, according to (4.4) and Remark 4.2,

$$(((\phi \times U) \otimes \lambda) \circ \widehat{\alpha}) \times (1 \otimes M) = ((\phi \otimes 1) \times (U \otimes \lambda)) \times (1 \otimes M)$$

is carried into

$$((\phi \otimes 1) \otimes_{\max} (1 \otimes M)) \times (U \otimes \lambda) = (\phi \otimes M) \times (U \otimes \lambda).$$

□

In what follows, we denote by X the $(A \otimes c_0(G/H)) \times_{\alpha\otimes \mathrm{lt}} G - A \times_\alpha H$ imprimitivity bimodule constructed by Green in [11], and we use X-Ind to denote the associated bijection of $\mathrm{Rep}(A \times_\alpha H)$ onto $\mathrm{Rep}((A \otimes c_0(G/H)) \times_{\alpha\otimes\mathrm{lt}} G)$.

Theorem 4.3. *Suppose that (G, H) is a discrete Hecke pair, that α is an action of G on a C^*-algebra A, and (ψ, T) is a covariant representation of (A, H, α) on some Hilbert space $\mathcal{H}_\psi$. Let $\mathcal{H} = X \otimes_{A\times_\alpha H} \mathcal{H}_\psi$, and let $\pi\colon A \times_\alpha G \to B(\mathcal{H})$ and $\nu\colon c_0(G/H) \to B(\mathcal{H})$ be such that $\pi\times\nu$ is the representation of $(A\times_\alpha G)\times_{\widehat{\alpha}|}(G/H)$ corresponding to X-Ind$(\psi\times T)$ under the isomorphism of Proposition 4.1. Then there exists a representation $(\psi, \overline{T})$ of (A, G, α) on $\mathcal{H}_\psi$ such that $\overline{T}|_H = T$ if and only if there exists a representation V of $\mathcal{H}(G, H)$ on $\mathcal{H}$ such that (ν, V) is a covariant pair and such that the range of V commutes with the range of π.*

In the special case where $G = N \rtimes Q$, S, and H are as in Section 2 and Corollary 3.3, there exists such a representation $(\psi, \overline{T})$ if and only if there exist a unitary representation $U\colon N/H \to B(\mathcal{H})$ and an isometric representation $W\colon S \to B(\mathcal{H})$ which, together with π and ν, satisfy conditions (i)–(iv) of Corollary 3.3.

Proof. By [12, Theorem 1], there exists such a representation $(\psi, \overline{T})$ if and only if there exists a representation $\phi\times U$ of $A\times_\alpha G$ such that X-Ind$(\psi\times T)$ is unitarily equivalent to $(\phi\otimes M)\times(U\otimes\lambda)$; by Proposition 4.1 these are precisely the representations $\phi\times U$ such that $\pi\times\nu$ is unitarily equivalent to the regular representation $(((\phi\times U)\otimes\lambda)\circ\widehat{\alpha})\times(1\otimes M)$. Thus the results follow by applying Theorem 3.2 and Corollary 3.3 to the maximal coaction $\delta = \widehat{\alpha}$ of G on $B = A \times_\alpha G$. □

In principle, whenever H is normal in N and N is normal in G (as occurs in the second part of Theorem 4.3), the question of extending representations from H to G is answered by applying [12, Corollary 4] twice: a representation of H extends to G if and only if it extends to N and the extension in turn extends to G. But in practice, the criteria for extending from N to G cannot be verified because the extension from H to N, when it exists, is not explicitly constructed in [12].

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School of Mathematics, The University of New South Wales, NSW 2052, Australia
E-mail address: astrid@unsw.edu.au

Department of Mathematics and Statistics, Arizona State University, AZ 85287-1804, USA
E-mail address: kaliszewski@asu.edu

School of Mathematical and Physical Sciences, University of Newcastle, NSW 2308, Australia
E-mail address: Iain.Raeburn@newcastle.edu.au