Chapter 3
Unforced Motions: Waves

Topics We Will Cover

1. Long Surface Gravity Waves
   - Wave
     - Refraction
     - Diffraction
     - Steepening
     - Breaking
2. Shallow and Deep Water Waves
3. Dispersion Relations, Phase & Group Velocity
4. Tides and Resonance
5. Potential and Relative Vorticity

Dispersion Relationship

In general wave frequency $\omega$ is related to wave number $k$ by the general dispersion relation.

The dispersion relationship for a plane 'shallow water' wave is given by

$$\omega = k\sqrt{gh}$$

In general wave frequency $\omega$ is related to wave number $k$ by the general dispersion relation.

$$\omega^2 = gk \tanh(kh)$$

Dispersion Relationship

For a 2D wave travelling in the x direction:

$$\frac{d^2 \eta}{dt^2} - gh \frac{d^2 \eta}{dx^2} = 0$$

Assumptions:
- No Coriolis
- No friction
- Constant density
- Shallow water/motion of water particles x-dir only

Try a wave like solution

$\eta = A \cos(kx - \omega t)$

$\omega^2 = ghk^2$

Dispersion relation

If we relax the shallow water approximation, so water particles can move in x and z direction:

$$\omega^2 = gk \tanh(kh)$$

Dispersion relation

But should be able to get back to shallow water dispersion relation for $h \ll \lambda$ …
Generalised Dispersion Relation

- **Deep water approximation**
  - If $h \gg \lambda$, $kh \gg 1$, $\tanh(kh) \approx 1$
  - $\omega^2 = gk$

Shallow water approximation
- If $h \ll \lambda$, $kh \ll 1$, $\tanh(kh) \approx kh$
  - $\omega^2 = gk^2h$

In the transition zone, need to use the full equation:
  - $\omega^2 = gk \tanh(kh)$

Phase Velocity

- **Phase Velocity**: The speed at which a particular phase of the wave (e.g. a crest) advances.
- $c = \frac{\omega}{k}$
  - $\omega = \frac{2\pi}{T}$
  - $k = \frac{2\pi}{\lambda}$

- The direction of propagation is perpendicular to the wave crest
When the wind blows during a storm waves are generated with many different periods. They 'add' together.

The movie shows two wave pulses, one is moving to the right, the other is moving to the left. They pass through each other without being disturbed, and the net displacement is the sum of the two individual displacements. It should also be mentioned that for this to work exactly the waves must be non dispersive (all frequencies travel at the same speed). If they were dispersive, then the waves would change their shape.

See http://www.kettering.edu/~drussell/Demos/superposition/superposition.html

The animation shows two sinusoidal waves travelling in the same direction. The phase difference between the two waves varies with time so that the effects of both constructive and destructive interference may be seen. First of all, notice that the sum wave (in blue) is a travelling wave which moves from left to right. When the two gray waves are in phase the result is large amplitude. When the two gray waves become out of phase the sum wave is zero.

The movie shows how a standing wave may be created from two travelling waves. If two sinusoidal waves having the same frequency (wavelength) and the same amplitude are travelling in opposite directions in the same medium then, using superposition, the net displacement of the medium is the sum of the two waves. When the two waves are 180° out-of-phase with each other they cancel, and when they are in-phase with each other they add together.
Wave Interaction

In the movie two waves with slightly different frequencies are travelling to the right. The resulting wave travels in the same direction and with the same speed as the two component waves. The "beat" wave oscillates with the average frequency, and its amplitude envelope varies according to the difference frequency.

Wave Groups

Waves that have ‘interacted’ can move in groups

Ocean-Surface Waves: the direction of propagation is perpendicular to the wave crests in the positive x direction.
Rossby Waves / Kelvin Waves (later!) the group velocity is not necessarily in the direction perpendicular to the wave crests.

Wave Groups

See Matlab examples

Group Velocity

- Group Velocity is the velocity at which a group of waves travels, & the velocity that wave energy propagates:
  \[ c_g = \frac{d\omega}{dk} \]

- Using the approximations for the dispersion relation it can be shown:
- Deep Water Group Velocity
  \[ \omega^2 = gk \quad h > \lambda/4 \]
- Shallow-water Group Velocity
  \[ \omega^2 = gk^2h \quad h < \lambda/11 \]
Group Velocity

- Group Velocity is the velocity at which a group of waves travels, & the velocity that wave energy propagates:

\[ c_g = \frac{d\omega}{dk} \]

Remember the Phase Velocity is given by \( c = \frac{\omega}{k} \)

- Using the approximations for the dispersion relation it can be shown:

- Deep Water Group Velocity

\[ \omega^2 = gk \quad h > \frac{\lambda}{4} \]

\[ c_g = \frac{g}{2\omega} = \frac{c}{2} \]

- Shallow-water Group Velocity

\[ \omega^2 = gk^2h \quad h < \frac{\lambda}{11} \]

\[ c_g = \sqrt{gh} \]

Phase Speed and Group Speed

Superposition of two sinusoidal travelling waves

Shows the difference between:

- Phase Speed (Cp) of the wave crests &
- Group Speed (Cg) of the wave envelope.

A wave packet propagating with the group velocity carries a plane wave with crest moving with the phase speed.

For long waves (kh<<1) the group velocity coincides with the phase speed.

\[ c_g = \sqrt{gh} \]

For short waves the group velocity is ½ the phase speed.

\[ c_g = \frac{g}{2\omega} = \frac{c}{2} \]

Wave Groups/Packets/Trains

- A group of deep-water waves moves at half the phase speed of the waves making up the group.
- E.g Watching deep water waves (at the back of a boat?!?) waves crests appear at the back of the wave train, move through the train, and disappear at the leading edge of the group.
- Each crest moves at twice the speed of the group.
Sorting (Dispersion) of Waves

Wave number $k$ is related to wave length $\lambda$

$$k = \frac{2\pi}{\lambda}$$

Bigger wavelength = small wave no.
But we know phase speed is related to frequency and wave number too.

$$c = \frac{\omega}{k} = \frac{g\lambda}{\sqrt{2\pi}}$$

Hence waves of different wavelengths starting from the same place will propagate at different rates, and hence will spread out or disperse.

A storm 3000km off shore, creates swell waves with wavelengths between 20 and 30 m. Estimate the time between the first and last sets of waves.

‘Clean’ Waves: Deep water waves

- The longer wavelength waves reach shore quicker than the shorter wavelength waves,
- Because speed depends on their wave length.
- Waves that have traveled a long distance are ‘sorted’ into wave packets, creating ‘clean’ swell.

Tides

- Tides are long (shallow water) gravity waves
- Produced by the gravitational pull of the sun, moon & other planetary bodies.
- Usually of period ~12 hrs or ~24 hrs
- Dominant period of 12 hrs 25 mins (1/2 lunar day)
Tides

Travel with a phase speed of \( c = \sqrt{gh} \)
Amplitude increases as \( h \) decreases (shelf, coast)
Propagation and Amplitude are affected by friction, rotation and bathymetry (shape and depth) of the oceans and seas.
Effects: Rise and fall in sea level and regular change in current speed and direction.

Tidal ranges can be extremely large and thereby dramatically influence coastal and estuarine conditions

Strength of the tidal force
- The Gravitational force along a straight line between 2 bodies is:
  \[ F_g = \frac{GM_1 M_2}{R^2} \]
  - \( G \) is the Universal gravitational constant
  - \( G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2 \)
  The Sun is bigger than the moon, but the moon is a lot closer, so the sun’s ‘tide generating force’ is only 46% of that of the moon.

- Centrifugal forces are the same everywhere ...
- Centrifugal forces are stronger closest to the sun/moon.
The joint influence of the sun and moon provide a variety of Principal Harmonic Constituents, the most important of which are:

<table>
<thead>
<tr>
<th>Designation</th>
<th>Name</th>
<th>Period (hrs)</th>
<th>Mag</th>
</tr>
</thead>
<tbody>
<tr>
<td>M2</td>
<td>Principal Lunar</td>
<td>12.42</td>
<td>1</td>
</tr>
<tr>
<td>S2</td>
<td>Principal Solar</td>
<td>12</td>
<td>0.46</td>
</tr>
<tr>
<td>K1</td>
<td>Luni-solar diurnal</td>
<td>23.9</td>
<td>0.584</td>
</tr>
<tr>
<td>O1</td>
<td>Principal lunar diurnal</td>
<td>25.8</td>
<td>0.415</td>
</tr>
</tbody>
</table>

- Tides can be expressed as a Sum of harmonic oscillations (periods).
- Each oscillation, (tidal constituent), has its amplitude, period and phase, which can be extracted from observations by harmonic analysis.
- Official tide tables use significantly more terms (e.g. the Australian National Tidal Facility uses 115 terms to produce the official Australian Tide Tables).

Spring tides occur when M2, and S2 reinforce.
Neap tides occur when M2, and S2 oppose. Successive springs occur about every 15 days.
Mixed tides occur when the relative magnitude of K1 and O1 are sufficiently large that they interfere with the semi-diurnal tides.
Spring tides occur when M2 and S2 reinforce. Neap tides occur when M2 and S2 oppose. Successive springs occur about every 14 days. Mixed tides occur when the relative magnitude of K1 and O1 are sufficiently large that they interfere with the semi-diurnal tides.

Since the earth rotates much faster than the progression of the bulge created by the moon’s gravitational attraction there is friction between the ocean and the seabed as the Earth turns out from underneath the ocean tidal bulges. This drags the ocean bulge in the eastward direction of the Earth’s rotation. Result is that ocean tides lead the Moon by about 10-degrees. This effect is known as Tidal Braking.

Slows the Earth’s rotation a tiny amount. The Day is getting gradually longer by 0.0023 seconds per century. See http://www.astronomy.ohio-state.edu/~pogge/Ast161/Unit4/tides.html

Co-range lines (lines of constant tidal range) run around amphidromic points in quasi-circular fashion. Co-phase lines (lines of constant phase, or lines which connect all places where high water occurs at the same time) emanate from amphidromic points like spokes of a wheel.

Tidal Currents

As the water moves in and out with the tide, currents are generated. Tidal currents are among the strongest currents in the world. Sydney Harbour circulation is dominated by tidal currents.
Tidal info on the web

- Official tide data for comparison:

Resonance

- The amplitude of a tide may be enhanced by resonance within bays if the length is appropriate.

At a distance $x=L$ sea level is given by:
$$\eta_{inc} = \eta_0 \cos(\omega t)$$
Where $\omega = 2\pi / \text{(Tidal period)}$ and $\eta_0 = \text{constant}$

Recall the wave equation (3.3)
$$\frac{d^2 \eta}{dt^2} - gh \frac{d^2 \eta}{dx^2} = 0$$

Assume a solution
$$\eta = A \cos(kx) \cos(\omega t)$$
$$\eta = \frac{A}{2} \left[ \cos(kx - \omega t) + \cos(kx + \omega t) \right]$$

From the equations of motion
$$\frac{du}{dt} = -g \frac{d\eta}{dx}$$
But at the coast, when $x=0$, $u_t=0$, hence
$$-g \frac{d\eta}{dx} = 0$$

At $x=L$ the sea level must match the tide level
$$\eta = A \cos(kL) \cos(\omega t) = \eta_0 \cos(\omega t)$$

$$A = \frac{\eta_0}{\cos(kL)}$$
\( \cos(kL) = 0 \) if \( kL = \frac{\pi}{2} \)

RECALL \( k = \frac{\omega}{c} \) & \( \omega = \frac{2\pi}{T} \)

\[ L = \frac{\pi}{2k} = \frac{\pi}{2\omega} = \frac{cT}{4} = \frac{\sqrt{ghL}}{4} \]

RECALL \( A = \frac{\eta L}{\cos(kL)} \) if \( \cos(kL) \approx 0 \) then \( A \) becomes infinite

Construct a bay that would show tidal resonance

---

Relative, planetary and potential Vorticity

We are familiar with conservation of angular momentum for a rigid body:

The angular velocity of a rigid body is changed by:
- A source or sink of angular momentum (e.g., friction)
- Changes in the body's moment of inertia (e.g., ice skater putting arms in and out)

The apparent angular velocity can change if we are in a rotating frame of reference.

In a fluid, it is not immediately apparent what the angular momentum of a fluid element is. The equivalent concept for a fluid is VORTICITY.

---

Bay of Fundy

Relative, planetary and potential Vorticity

Two types of vorticity:
- **Planetary** Vorticity (because the planet is rotating)
- **Relative** Vorticity (spinning on its own axis)

Planetary vorticity is usually much bigger than relative vorticity \( f \gg \zeta \)

The sum of planetary and relative vorticity is called the absolute vorticity.
Relative Vorticity $\zeta$

Vorticity is actually a vector quantity, but we are usually only concerned with the vertical component. The vertical component of the relative vorticity is:

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Imagine a ball fixed at a point in the flow. The spin of the ball is related to the vorticity. See [http://www.math.umn.edu/~nykamp/m2374/readings/divcurl/](http://www.math.umn.edu/~nykamp/m2374/readings/divcurl/)

Planetary Vorticity $f$

(2 $\pi$ radians per day)

$\Omega = 2\pi \sin(\phi)$

- At the poles $f=2\Omega$ ($\phi=90$)
- At the equator $f=0$ ($\phi=0$)

Relative & planetary vorticity $\zeta$

$$f = 2\pi\sin(\phi)$$

Shear: Change in velocity with distance:

1. What is the relative vorticity?
2. At 30°S what is the planetary vorticity?
Friction causing shear

\[ \zeta = \frac{dv}{dx}, \quad \zeta = \frac{dv}{dx} > 0 \]

\( v = 0 \) at the wall,
\( v > 0 \) away from the wall.
Creates positive spin \( \zeta > 0 \)

Sign of Vorticity

- Counter-clockwise = positive
- Clockwise = negative
- Both planetary and relative vorticity

Gulf Stream Velocity / Relative Vorticity

Assume \( \frac{du}{dy} = 0 \)

More +ve relative vorticity

Less -ve relative vorticity
Absolute Vorticity

Absolute amount of vorticity is conserved (as long as water depth remains constant)

Absolute Vorticity = \( f + \zeta \)

Individual numbers can go up and down, but the \textit{sum} is constant

\( f > \zeta \), Absolute Vorticity is \textit{usually} positive in the N. Hem (\( f>0 \)) and usually negative in the S. Hem. (\( f<0 \)).

Angular momentum tends to be conserved as columns of water change latitude.

This causes changes in relative vorticity of the columns

Absolute amount of vorticity is conserved. Change in latitude = change in spin.

Potential Vorticity \( Q \)

In the absence of friction (and other forces) it is actually Potential Vorticity (not absolute vorticity) that is conserved

We must include height (+ve) in the vorticity conservation equation

\[
Q = \frac{f + \zeta}{h}, \quad \frac{d}{dt}(Q) = 0
\]

Hence as a parcel moves from one latitude to another there must be a change in either \textit{thickness (height)} or \textit{spin (relative vorticity)}.

Consider a skater: With arms outstretched her spin is \( f \).

As she brings her arms in, her height increases.

Potential Vorticity remains constant

\[
Q_j = \frac{f}{h_j} = Q_i = \frac{f + \zeta}{h_j}, \quad k_j > k_i, \quad \zeta_j = f \frac{h_j}{k_j}, \quad \zeta_j > 0
\]
Change in Relative Vorticity
Column Stretching

Production of relative vorticity by changes in the height of a fluid column. As the vertical fluid column moves from left to right:
- vertical stretching reduces the moment of inertia of the column;
- causing it to spin faster.
We are not considering the effect of planetary vorticity here.

Topographic Steering
Barotropic flow over a sub-sea ridge is deflected equatorward to conserve potential vorticity. As the depth decreases,
- $\zeta + f$ must also decrease,
- which requires that $f$ decrease
- flow is turned toward the equator.

Vorticity problem
This eddy is located at 30°N latitude. It is 50 km across.
- What is the planetary vorticity?
- What is the absolute vorticity?

If this eddy drifts northward to Greenland, at 60°N.
- What is the planetary vorticity?
- What is the absolute vorticity?
- What is its relative vorticity?
- Which way is it spinning?

Summary
- Planetary Vorticity $f$ (rotating because the planet is rotating)
- Relative Vorticity (spinning on its own axis)
- Potential Vorticity includes height & is conserved.
- Flow is deflected around 'bumps' to conserve potential vorticity.
**Vorticity questions**

Water parcels move around the ocean, their spin or rotation changes to conserve angular momentum. How will the spin of each of the following eddies change - will they spin faster or slower?

Hint: write down the signs of the relative and planetary vorticities, and remember that the absolute vorticity is conserved.

A) An anti-cyclonic eddy in the N. Hemisphere begins to drift poleward.
B) An anti-cyclonic eddy in the S. Hemisphere begins to drift poleward.
C) A cyclonic eddy in the N. Hemisphere begins to drift poleward.
D) A cyclonic eddy in the S. Hemisphere begins to drift poleward.
E) A non-rotating parcel eddy in the N. Hemisphere begins to drift poleward.
F) A non-rotating parcel eddy in the S. Hemisphere begins to drift poleward.

**Potential vorticity questions**

For each of the following cases, determine the possible responses of the water parcel.

A) An anti-cyclonic eddy in the N. Hemisphere drifts over a seamount.
B) An anti-cyclonic eddy in the S. Hemisphere drifts over a trench.
C) A cyclonic eddy in the N. Hemisphere drifts over a trench.
D) A cyclonic eddy in the S. Hemisphere drifts over a seamount.
E) A non-rotating parcel eddy in the N. Hemisphere drifts over a seamount.
F) A non-rotating parcel eddy in the S. Hemisphere drifts over a seamount.

**Revisiting the problem**

You find that the eddy you tracked to Greenland is now actually in 4 km deep water, while the water was 2 km deep off of South Carolina.

What was the potential vorticity at 30°N?
What is the potential vorticity at 60°N?
What is the new planetary vorticity?
What is the new relative vorticity?
So what is the eddy doing now?

**Relative Vorticity Calculation**

A cold core eddy formed off the coast of the US is 150 km wide with the following speeds:
- the north end is 1.2 m/s,
- the south end 1.1 m/s,
- the west side 1.1 m/s,
- the east side 1.2 m/s.

What is the relative vorticity of this eddy?
Questions