Chapter 3
Unforced Motions: Waves

Topics We Will Cover
1. Long Surface Gravity Waves
2. Wave
   • Refraction
   • Diffraction
   • Steepening
   • Breaking
3. Dispersion Relationships
4. Shallow and Deep water waves
5. Phase Velocity & Group Velocity
6. Tides and Resonance
   Potential and Relative Vorticity

Review of Chapter 2
1. Introduction to the Equations of Motion
2. The Vertical Equation
   1. Equation of State
   2. Buoyancy Effects
3. The Horizontal Equations
4. Rotation – Coriolis Force
5. Geostrophic Balance
6. Thermal Wind Balance
7. Taylor Sheets, Columns and Blocking
8. Friction and Incompressibility
Hydrostatic Balance

This \( \frac{dw}{dt} \) is much smaller than gravity, hence, gravity cannot be balanced by \( \frac{dw}{dt} \), thus

\[
\frac{dp}{dz} = \rho g
\]

Generally most large scale ocean and atmospheric circulation of period > 1 day in which the vertical motion is weak, satisfies the hydrostatic balance. This relationship does not hold in vigorous systems of smaller horizontal scales such as convection.

Coriolis Force

- Rotational speed of the Earth varies with latitude \( \Phi \)
- A Coriolis parameter, \( f \) can be defined: \( f = 2\Omega \sin \Phi \), \( \Omega = 7.3 \times 10^{-5} \) sec\(^{-1} \)

The acceleration due to the Coriolis Force is:
- \( fv \) in the x-direction, \( -fu \) in the y-direction
- Acts to the Left (right) in the Southern (northern) hemisphere.
Equations of Motion

Horizontal Equations:

**Acceleration = Pressure Gradient force + Coriolis**

\[
\frac{du}{dt} = -\frac{1}{\rho_0} \frac{dp}{dx} + f v \\
\frac{dv}{dt} = -\frac{1}{\rho_0} \frac{dp}{dy} - f u
\]

Vertical Equation:

**Pressure Gradient force = Gravitational Force**

\[
\frac{dp}{dz} = \rho g
\]

In a Barotropic Ocean....

For a barotropic ocean, \( \rho = \rho_0 \) and \( p = p_0 + p' \) with \( p' = \rho_0 g \eta \) the horizontal equations are further simplified:

\[
\frac{du}{dt} = -g \frac{d\eta}{dx} + f v \\
\frac{dv}{dt} = -g \frac{d\eta}{dy} - f u
\]

i.e. acceleration in \( x \) and \( y \) directions depends on sea surface height and rotation.
Geostrophic Balance

Geostrophic Currents

Geostrophic Eddy
(Northern Hemisphere)
Thermal Wind Balance

The Vertical Structure of $u$ and $v$ is related to horizontal density gradients

\[ v = \frac{1}{\rho f} \frac{dp}{dx} \]
\[ u = -\frac{1}{\rho f} \frac{dp}{dy} \]
\[ \frac{dp}{dz} = \rho g \]

\[ \frac{dv}{dz} = \frac{g}{\rho_o f} \frac{d\rho}{dx} \]
\[ \frac{du}{dz} = -\frac{g}{\rho_o f} \frac{d\rho}{dy} \]

i.e horizontal density gradients ($t, s$) can explain vertical velocity (wind) profiles

Water Waves

- Transfers a disturbance from one point of a material to another
- No net overall motion to the material >> Energy is transported through the water without any transport of the water itself
- Oscillatory Particle movement as a wave propagates
- No change to waveform
- Constant speed
Characteristics of a Waves

- **Frequency**: Number of wave crests passing a point each second.
- **Period**: Time required for a crest at point A to reach point B.

Continuity

Conservation of Mass states that mass must be conserved. That is, any imbalance between convergence and divergence in the three spatial directions must create a local compression or expansion of the fluid.

Mathematically, the statement takes the form:

\[
\frac{\partial p}{\partial t} + \frac{\partial pu}{\partial x} + \frac{\partial pw}{\partial y} + \frac{\partial pw}{\partial z} = 0
\]

Assume density is constant:

\[
\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0
\]
Near incompressibility of sea water means that as sea level changes, flow into a region must be balanced by flow out. In a barotropic ocean:

\[
\frac{d\eta}{dt} + \frac{d(hu)}{dx} + \frac{d(hv)}{dy} = 0
\]

If \(v=0\) & \(h=\text{const}\), mass conservation requires:

\[
\frac{d\eta}{dt} + h \frac{du}{dx} = 0
\]

As sea level changes, water must flow in or out to compensate; divergence and convergence.

The relationship between wavelength and depth in determines wave type.

**Deep water wave**
- \(h > \lambda/2\)
- Orbital partial paths

**Shallow water wave**
- \(h \leq \lambda/20\)
- Elliptical particle paths

**Transitional wave**
- \(\lambda/20 \leq h < \lambda/2\)
- Elliptical particle paths
Shallow water waves

- Waves propagate along the air sea interface.
- Assume that the flow is 2-dimensional with waves travelling in the x-direction.
- Assume the Coriolis force and viscosity can be neglected. (If we retain rotation, we get Kelvin waves.)
- Density differences within the ocean are SMALL compared to the difference in density between air and sea.
- Thus we can assume a BAROTROPIC ocean, i.e. $\rho$=constant
- We also assume a constant depth $h$
- We can write the equations of motion as
- Which can be combined to form the wave equation

$$\frac{d^2 \eta}{dt^2} - gh \frac{d^2 \eta}{dx^2} = 0$$

A solution to the wave equation is of the form

$$\eta = A \cos(kx - \omega t)$$

Showing the sea-surface elevation $\eta$ of a wave travelling in the x direction

By substitution in the wave equation we can show that

$$\frac{\omega^2}{k^2} = gh$$

Frequency: $\omega$=2$\pi$/T
Wavenumber: $k$=2$\pi$/\Lambda

Period (T) time for a crest at point A to reach point B

Frequency ($\omega$): Number of wave crests passing a point each second
Wavenumber (k): is a wave property inversely related to wavelength, (units = m$^{-1}$). Wavenumber is the spatial analogue of frequency i.e. the number of times a wave has the same phase per unit of space.
Dispersion Relationship

- In general wave frequency $\omega$ is related to wave number $k$ by the general dispersion relation.
- The dispersion relationship for a plane ‘shallow water’ wave is given by

$$\omega = k\sqrt{gh}$$

Generalised Dispersion Relation

In the derivation for the shallow water wave, we assumed that velocity doesn’t change with depth and motion of the water is essentially horizontal. This assumption breaks down if we are do not have a shallow water wave. We need to add extra information (this is covered in MATH 3261)
Generalised Dispersion Relation

For a general wave the dispersion relation is actually …

\[ \omega^2 = gk \tanh(kh) \]

- Deep water approximation
  - If \( h \gg \lambda, \ kh \gg 1 \), \( \tanh(kh) \approx 1 \)
  - \( \omega^2 = gk \ h > \lambda / 4 \)

- Shallow water approximation
  - If \( h \ll \lambda, \ kh \ll 1 \), \( \tanh(kh) \approx kh \)
  - \( \omega^2 = gk^2h \ h < \lambda / 11 \)

For a general wave the dispersion relation is actually ...

\[ \omega^2 = gk \tanh(kh) \]

Phase Velocity

- Phase Velocity: The speed at which a particular phase of the wave (e.g. a crest) advances.

\[ c = \frac{\omega}{k} \quad \omega^2 = gk \tanh(kh) \]

- \( k = \frac{2\pi}{\lambda} \)

- The direction of propagation is perpendicular to the wave crest (+ve x dn)

\[ c = \frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi h}{\lambda}\right) \]
Phase Velocities

\[ c = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh(kh)} \]

A complex equation…. so we make approximations

Deep water waves: \( h > \frac{\lambda}{2} \):

\[ c = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} = \frac{g}{\omega} \]

Bigger wavelength = faster waves, thus **DISPERSE**

Shallow water waves: \( h < \frac{\lambda}{20} \):

\[ c = \sqrt{gh} \]

Slow waves

Types of water waves

Shallow water waves \( h \leq \frac{\lambda}{20} \):
Speed depends on the water depth
- waves near coast,
- tides [\( \lambda \sim 1000\text{’s km} \)],
- tsunamis [\( \lambda \sim 100\text{’s km} \)]

Deep water waves \( h > \frac{\lambda}{2} \):
Speed does not depend on the depth of the water. Only depends on the wavelength.
- (e.g. swell waves)
3.2 Refraction, Diffraction and Shoaling

Wave Refraction:

Wave crests tend to become parallel to the shore as the wave moves inshore...

Because $c = \sqrt{gh}$, c decreases as h decreases

$$C_j = \sqrt{gh_j} > C_i = \sqrt{gh_i}$$

Wave energy over a shallow submerged ridge is focussed on the headland.

Increased energy per unit length of wave crest as the height of the wave increases.

Waves refracted by the shallow water deliver lower levels of energy inside the bay.

Ray divergence spreads the energy over a larger volume, decreasing energy per unit length as the height increases.
Wave Diffraction

- The bending of waves when they interact with an obstacle.
- Effect is more pronounced when the wave length is similar to the size of the obstacle.
- When waves reach a narrow slit the diffracted wave resembles a circular wave with centre at the slit.
- A wave travels in a straight line when the size of the slit is much larger than the wavelength.

Wave Diffraction at the northern entrance to the Panama Canal, as viewed from Google Earth.
3.2.2 Energy Flux - Shoaling

Waves steepens as depth decreases: Flux of energy remains constant until the wave breaks.

Energy Flux for a surface wave \( F = cE \)

Where \( E \) (the energy per m\(^2\)) is:
\[
E = \frac{1}{2} \rho_0 g A^2
\]

\( \eta = A \cos(kx - \omega t) \)

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Energy Flux has to stay the same
(Conservation of energy)

In absence of friction and wave breaking...

\( F_i = c_i E_i = F_j = c_j E_j \)

Hence the amplitude inshore is greater than offshore, i.e. the wave steepens as the speed decreases.

\[
A_j^2 = \left( \frac{h_i}{h_j} \right)^{1/2} A_i^2
\]
3.2.3 Wave Breaking

Observations show that a wave will steepen and break if the amplitude gets big enough...

\[ A \geq \lambda/12 \]

For tsunamis, this criteria will not be met, as \( L \) is very large... hence we use

\[ A \geq 0.8 \, h \]

Breakers
Generation of Wind Waves and Swell

The stronger the wind, The longer it blows, & the bigger the ‘fetch’ = the bigger the waves!

The global distribution of wind speed and wave height.

Strong winds in the southern ocean (Roaring Forties) generate huge seas, (large fetch and duration)
Summary

Wave Information WWW

- http://cdip.ucsd.edu/
- www.swellnet.com.au
- www.realsurf.com
- Stewart Chapter 16
Next Lecture

• Tides
• Vorticity – Conservation
• Internal Waves
• Resonance ¼ Wave Oscillators