Introduction to Oceanography and Meteorology
MATH2240

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1 Introduction

The oceans and atmosphere are crucial to the existence and behaviour of all living organisms on our planet. Changes in weather, due to large scale events such as El Niño, can lead to floods, droughts and fires with devastating effects on fisheries, crop production and livestock. At smaller scales, we observe the passage of high and low pressure systems and might ask why the air does not rush from one region to the other. The passage of fronts are accompanied by large changes in temperature and wind – why? At the beach we observe tides and waves and if unlucky – tsunamis! These waves can travel as fast as a jumbo jet and can have devastating effects on coastal regions before evacuation can occur.

In this course we will examine some of the fundamental causes and dynamics of these phenomena building upon common sense experience and elementary first year mathematics. Simple thought and laboratory experiments will be used and students are encouraged to seek further examples from the internet. We will be taking a field trip to Maroubra Beach to investigate different types of waves and coastal processes. Many of the phenomena will be illustrated using examples from the oceans, although it should be emphasised that the dynamics of the oceans and atmosphere are very similar.

1.1 Course Convenor

The course convener is Dr Moninya Roughan Her contact details are:

Room: RC-4063, Red Centre,

Phone: 9385 7067

Email: mroughan@unsw.edu.au

1.2 Book References

B. Cushman-Roisin: Introduction to Geophysical Fluid Dynamics, Prentice-Hall, 1994

R. H. Stewart Introduction to Physical Oceanography (can be downloaded from course web page)


Ocean Circulation, The Open University ISBN 0 08 036369;

J. T. Houghton The Physics of Atmospheres;
B. Crowder The Wonders of the Weather;

A rather deeper and more mathematical treatment of a whole range of geophysical processes is given in

A. Gill: Atmosphere-Ocean Dynamics, Academic, 1982

1.3 Web Sites

This list is not exhaustive, but it is a good start.

http://podaac.jpl.nasa.gov/tecd.html
http://www.ncdc.noaa.gov/onlineprod/drought/xmgr.html
http://www.bom.gov.au

Oceanography Notes from other sources
http://sealevel.jpl.nasa.gov/education/edudoc.html
http://oceanworld.tamu.edu
http://oceanworld.tamu.edu/students/waves/index.html
http://oceanworld.tamu.edu/students/currents/index.html

Australian Oceanographic Institutes
http://www.mth.uea.ac.uk/ocean/vl/australasia.html

Societies and Organisations
http://www.mth.uea.ac.uk/ocean/vl/societies.html

Data
http://meso-a.gsfc.nasa.gov/rsd/

UNSW
www.cedl.unsw.edu.au
1.4 Introduction to The Ocean and The Atmosphere

71% of the earth’s surface is surrounded by ocean, and only 29% of it is land. The main oceans are the Pacific (46%), Atlantic (23%) and Indian (20%) as well as the Southern Ocean and the Arctic Ocean. The remainder is made up of coastal seas. On average the oceans are ~ 4km deep. Most continents are surrounded by a continental shelf region ~ 100 – 200 m deep. It is in this region that the majority of the world’s productivity occurs.

1.5 Properties of the Ocean

1.5.1 Temperature in the Ocean

Temperature is a measure of the heat content of water and at the surface varies from $T = 0 \, ^\circ C$ at poles to $T = 28 \, ^\circ C$ at the equator. At abyssal depths a nearly constant temperature of $0 - 4 \, ^\circ C$ pertains. Since temperature is dependent upon pressure, we sometimes use potential temperature $\theta$ which is the temperature of a water mass if it were brought to a reference pressure level (usually 1 atmosphere; the surface). Temperature T is sometimes called in situ temperature and is warmer than $\theta$: since water is slightly compressible, a sample brought from depth will expand then cool. Potential temperature $\theta$ is calculated from temperature, salinity and pressure using a complicated empirical formula. In situ temperature T is that which is measured locally (e.g. the thermometer reading at depth), potential temperature $\theta$ is the temperature this water would have if it were at sealevel pressure.

Temperature is important because it reflects the amount of heat held and transported by the ocean. The temperature of the ocean is primarily influenced by the heating at the air-sea interface and varies both horizontally and vertically in the ocean.

1.5.2 Salinity in the Ocean

Total dissolved solids (mainly sodium chloride, or Table salt) - About 3.5% by weight (average seawater) - Usually expressed as 35 psu (practical salinity units, psu, or no units at all) - Varies geographically according to Evaporation, precipitation, rivers, ice formation and ice melt.

1.5.3 Pressure

Ocean pressure is the weight of seawater per unit area (force per unit area). It is mainly a function of depth. Pressure in the ocean increases at a rate of about 1 atmosphere per 10 m of water. Or pressure increases by 1 dbar per 1 m of water.
1.5.4 Density

Density ($\rho$) is the mass of water per unit volume, and is measured in gm/cc or in kg m$^{-3}$ (SI) and depends on water temperature $T$ (colder water is denser), salinity $S$ (saltier water is denser) and pressure $p$ (water is compressible). The relationship between $\rho$ and $(S, T, p)$ is very complicated and is referred to as ‘The Equation of State’. However, for regions where $T$ and $S$ vary little, we may assume a linear equation of state, namely

$$\rho \approx \rho_0 (1 - \alpha T + \beta S)$$

where $\alpha$ and $\beta$ are (nearly constant) expansion coefficients for temperature and salinity, and $\rho_0$ is a reference density. Density ($\rho$) in the ocean is affected by pressure for two reasons: (i) seawater is compressible (the ocean’s weight can squash a water parcel at depth into a smaller volume), and (ii) because $\rho$ depends on temperature which is itself affected by pressure (as described above). It is therefore useful to distinguish between in situ density which is the density of water in its local environment, and potential density which is the density this water would have at some reference depth, normally taken to be the sea surface. In summary, in situ density is a function of local $(T, S, p)$ whereas potential density is corrected for pressure effects, so depends on $(\theta, S, p = p_{ref})$, where $p_{ref}$ is normally 0 (i.e. atmospheric pressure).

At the sea surface, density varies from around 1020 to 1030 kg m$^{-3}$ i.e. by less than 1%. Thus it is often convenient to subtract out the 1000 kg m$^{-3}$ and deal with the residual. This quantity is referred to as $\sigma_t$ (‘sigma–t’), where

$$\sigma_t \equiv \rho(S, \theta, 0) - 1000$$

and reference pressure $p$ has been taken as zero (i.e. atmospheric pressure).

Density depends on salinity, temperature and pressure, and generally

- increases with increasing salinity
- increases with decreasing temperature

Seawater density ranges from $\sim 1021 - 1070$ kg m$^{-3}$, the average density is 1025 kg m$^{-3}$. Density increases with pressure, as the pressure force squashes water into a smaller volume. Generally for stability, less dense water overlies more dense water.

1.6 Properties of the Atmosphere

1.6.1 Composition

The earth’s atmosphere is a very thin layer that surrounds a very large planet. The constituents of the atmosphere by mass are N2 (75%), O2 (23.2%), and others such as argon
and CO2 (1.3%) Water vapor, also exists in small amounts, which range from $\sim 0\%$ over the desert to $\sim 4\%$ over the oceans. Water vapor is important to weather production since it exists in gaseous, liquid, and solid phases and absorbs radiant energy from the earth.

1.6.2 Temperature

Based on temperature, the atmosphere is divided into four vertical layers: the troposphere, stratosphere, mesosphere, and thermosphere. The air at the surface up to around 10 km is called the troposphere. The troposphere is very well mixed, because the air near the surface is warmer than the air above it. So, like a pot of boiling water, the warmer air is always rising. The reason it is warmer at the surface is simple. The air is warmed by radiation emitted by the Earth. The further away from the surface the air moves, the less radiation there is to absorb.

From 10 – 20 km the atmosphere is stable. This region is called the tropopause. From 20 to about 50 kilometers is the stratosphere. In this region the air actually warms with height. Ozone is concentrated in this part of the atmosphere and it absorbs ultraviolet light from the Sun. More light is absorbed at higher altitudes compared to the lower stratosphere, so the temperature increases.

But at 50 km, the temperature levels out again in a region called the stratopause. At about 55 km, the mesosphere begins. In the mesosphere, the temperature decreases with height again, because there is very little ozone to warm up the air. Like the troposphere, the mesosphere is well mixed because the warmer air below is always rising.

Finally, the mesopause divides the mesosphere from the thermosphere, which is the section of the atmosphere higher than 90 km. In this region, the temperature increases again. This time, it is molecular oxygen that causes the temperature increase. The oxygen absorbs light from the Sun, and since there is very little air in the thermosphere, just a little absorption can go a long way. This is just an average profile of the atmosphere. The arctic region will have a similar profile to the tropics, but the heights of each layer and the actual temperatures will be different.

1.6.3 Vertical Structure - Troposphere

The troposphere extends from the earth’s surface to an average of 12 km and the pressure ranges 1000 to 200mb. The temperature generally decreases with increasing height up to the tropopause (top of the troposphere); this is near 200mb or 12 km. Temperature range $\sim 15^\circ$C (surface) to $\sim -57^\circ$C at the tropopause. The layer ends at the point where temperature no longer varies with height. This area, known as the tropopause, marks the transition to the stratosphere. Winds increase with height up to the jet stream and the moisture concentration decreases with height up to the tropopause. The air is much drier above the tropopause, in the stratosphere. The sun’s heat that warms the earth’s surface is transported upwards largely by convection and is mixed by updrafts and downdrafts.
1.6.4 Water Vapour

Most of the water vapor in the atmosphere comes from the oceans. Most of the precipitation falling over land finds its way back to oceans. About two-thirds returns to the atmosphere via the water cycle. Oceans not only act as an abundant moisture source for the atmosphere but also as a heat source and sink (storage). The exchange of heat and moisture has profound effects on atmospheric processes near and over the oceans. Ocean currents play a significant role in transferring this heat poleward. Major currents, such as the northward flowing Gulf Stream, transport tremendous amounts of heat poleward and contribute to the development of many types of weather phenomena. They also warm the climate of nearby locations. Conversely, cold southward flowing currents, such as the California current, cool the climate of nearby locations.

1.6.5 Atmospheric Pressure

Atmospheric pressure is the pressure exerted by the air in the column above that height. Atmospheric pressure decreases with height above the earth, but the mean pressure at sea level is 1040-970hPa. Mean sea level pressure (MSLP) is the pressure at sea level or (when measured at a given elevation on land) the station pressure reduced to sea level assuming an isothermal layer at the station temperature. This is the pressure normally given in weather reports on radio, television, and newspapers or on the Internet. The reduction to sea level means that the normal range of fluctuations in pressure is the same for everyone. The pressures which are considered high pressure or low pressure do not depend on geographical location. This makes isobars on a weather map meaningful and useful tools.

1.6.6 Density in the Atmosphere

Density in the atmosphere is the number of air molecules in a given volume. As pressure decreases, density decreases as nothing squeezing the molecules together. Atmospheric pressure is the amount of force exerted on the earth's surface by a pile of air molecules. Air density decreases with increasing height. The density of air at sea level is about 1.2 kg m$^{-3}$ (1.2 g/L). Natural variations of the barometric pressure occur at any one altitude as a consequence of weather. This variation is relatively small for inhabited altitudes but much more pronounced in the outer atmosphere and space due to variable solar radiation. The atmospheric density decreases as the altitude increases. This variation can be approximately modelled using the barometric formula. More sophisticated models are used by meteorologists and space agencies to predict weather and orbital decay of satellites. The average mass of the atmosphere is about 5,000 trillion metric tons with an annual range due to water vapor of $1.2 - 1.5 \times 10^{15}$kg depending on whether surface pressure or water vapor data are used.
1.7 Distinctions between the Atmosphere and Oceans

There are many similarities between the motions that occur in the atmosphere and the ocean, however there are notable scale disparities. Figure 1 shows some of the length, velocity and time scales in the ocean and the atmosphere.

<table>
<thead>
<tr>
<th>Phenomenon</th>
<th>Length Scale $L$</th>
<th>Velocity Scale $U$</th>
<th>Time Scale $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Atmosphere:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Microturbulence</td>
<td>10–100 cm</td>
<td>5–50 cm/s</td>
<td>few seconds</td>
</tr>
<tr>
<td>Thunderstorms</td>
<td>few km</td>
<td>1–10 m/s</td>
<td>few hours</td>
</tr>
<tr>
<td>Sea breeze</td>
<td>5–50 km</td>
<td>1–10 m/s</td>
<td>6 hours</td>
</tr>
<tr>
<td>Tornado</td>
<td>10–500 m</td>
<td>30–100 m/s</td>
<td>10–60 minutes</td>
</tr>
<tr>
<td>Hurricane</td>
<td>300–500 km</td>
<td>30–60 m/s</td>
<td>Days to weeks</td>
</tr>
<tr>
<td>Mountain waves</td>
<td>10–100 km</td>
<td>1–20 m/s</td>
<td>Days</td>
</tr>
<tr>
<td>Weather patterns</td>
<td>100–5000 km</td>
<td>1–50 m/s</td>
<td>Days to weeks</td>
</tr>
<tr>
<td>Prevailing winds</td>
<td>Global</td>
<td>5–50 m/s</td>
<td>Seasons to years</td>
</tr>
<tr>
<td>Climatic variations</td>
<td>Global</td>
<td>1–50 m/s</td>
<td>Decades and beyond</td>
</tr>
<tr>
<td><strong>Ocean:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Microturbulence</td>
<td>1–100 cm</td>
<td>1–10 cm/s</td>
<td>10–100 s</td>
</tr>
<tr>
<td>Internal waves</td>
<td>1–20 km</td>
<td>0.05–0.5 m/s</td>
<td>Minutes to hours</td>
</tr>
<tr>
<td>Tides</td>
<td>Basin scale</td>
<td>1–100 m/s</td>
<td>Hours</td>
</tr>
<tr>
<td>Coastal upwelling</td>
<td>1–10 km</td>
<td>0.1–1 m/s</td>
<td>Several days</td>
</tr>
<tr>
<td>Fronts</td>
<td>1–20 km</td>
<td>0.5–5 m/s</td>
<td>Few days</td>
</tr>
<tr>
<td>Eddies</td>
<td>5–100 km</td>
<td>0.1–1 m/s</td>
<td>Days to weeks</td>
</tr>
<tr>
<td>Major currents</td>
<td>50–500 km</td>
<td>0.5–2 m/s</td>
<td>Weeks to seasons</td>
</tr>
<tr>
<td>Large-scale gyres</td>
<td>Basin scale</td>
<td>0.01–0.1 m/s</td>
<td>Decades and beyond</td>
</tr>
</tbody>
</table>

Figure 1: Length, Velocity and Time Scales in the earth’s atmosphere and oceans (Cushman-Roisin and Beckers, 2007)

There are a number of oceanic processes that are caused by the presence of lateral boundaries (landmasses) that do not occur in the atmosphere. Also, Atmospheric motions often depend on the concentration of moisture in the atmosphere. There are differences in forcing mechanisms, mainly thermodynamics in the atmosphere, while in the ocean, gravity (tides) and winds also play a role (which in turn are related to the thermodynamics of the atmosphere!). Furthermore there are differences in terminology and notation. In meteorology it is common to refer to winds by the direction of origin (e.g. a northerly) however in oceanography we are more interested in the direction of travel, e.g. a southward current (which originates in the north!). Be careful!!
1.8 Units Summary:

Always calculate quantities using kg, meters, seconds and Pascals and the answers will be in one of the units below.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Units</th>
<th>Units</th>
<th>Conversion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td></td>
<td>seconds</td>
<td>s</td>
<td>1 day (\approx 100000 s = 10^5 s)</td>
</tr>
<tr>
<td>Mass</td>
<td></td>
<td>kilograms</td>
<td>kg</td>
<td></td>
</tr>
<tr>
<td>Length</td>
<td></td>
<td>meters</td>
<td>m</td>
<td></td>
</tr>
<tr>
<td>Velocity</td>
<td>(u, v, w)</td>
<td>meters per sec</td>
<td>ms(^{-1})</td>
<td>2.78 (\text{ms}^{-1}=10 \text{ km h}^{-1})</td>
</tr>
<tr>
<td>Pressure</td>
<td>(p)</td>
<td>Pascals (force per unit area)</td>
<td>Pa</td>
<td>1 hectopascal (=100 \text{ Pa})</td>
</tr>
<tr>
<td>Salinity</td>
<td>(S)</td>
<td>kg salt per 1000 kg water</td>
<td>psu</td>
<td></td>
</tr>
<tr>
<td>Temperature (ocean)</td>
<td>(T)</td>
<td>Degrees Centigrade</td>
<td>^\circ\text{C}</td>
<td>(T (^\circ\text{C}) = T (\text{K}) - 273)</td>
</tr>
<tr>
<td>Temperature (atmos)</td>
<td>(T)</td>
<td>Kelvin</td>
<td>K</td>
<td></td>
</tr>
<tr>
<td>Density</td>
<td>(\rho)</td>
<td>kg per unit volume</td>
<td>kg m(^{-3})</td>
<td></td>
</tr>
<tr>
<td>Density</td>
<td>(\sigma)</td>
<td>&quot;</td>
<td>&quot;</td>
<td>(\sigma = \rho - 1000)</td>
</tr>
<tr>
<td>Coriolis Parameter</td>
<td>(f = 2 \Omega\ sin(\text{latitude}))</td>
<td>1/seconds</td>
<td>s(^{-1})</td>
<td></td>
</tr>
<tr>
<td>Windstress</td>
<td>(\tau)</td>
<td>Pascals (force per unit area)</td>
<td>Pa</td>
<td>(1 \text{ Pa} = 1 \text{ N m}^{-2}) (1 \text{ N} = 1 \text{ kgm s}^{-2})</td>
</tr>
<tr>
<td>Transport</td>
<td>(\text{Sv})</td>
<td>Sverdrups</td>
<td>m(^3\text{s}^{-1})</td>
<td>(1 \text{ Sv} = 1 \times 10^6 \text{ m}^3\text{s}^{-1})</td>
</tr>
</tbody>
</table>

1.9 Coordinate system and notation

In oceanography the Cartesian \((x, y, z)\) coordinate system is used to denote position in space. That is, \(x\) denotes east-west position, \(y\) north-south position, and \(z\) depth. Standard units are in metres. Position in the horizontal is also often quoted in degrees longitude and latitude. The corresponding velocity of water circulating in the ocean is denoted by \((u, v, w)\), where \(u\) is the east-west speed, \(v\) the north-south speed, and \(w\) the vertical velocity. Note that by convention, positive directions are, respectively, eastwards, northwards, and downwards.
1.10 Gradients and derivatives

In most applications in oceanography the gradient terms in equations, such as

\[ \frac{dT}{dx} , \quad \frac{d\rho}{dt} \quad \text{etc.} \]

can be approximated as follows:

\[ \frac{dT}{dx} \approx \frac{\Delta T}{\Delta x} \]

where \( \Delta \) means the ‘change in’ a certain property (temperature, time, and so on). This is generally valid for ‘small’ \( \Delta x \).
2 The Equations of Motion

To understand the circulation of the oceans and atmosphere it is necessary to examine the underlying equations which govern their motion. In the following, the equations are all derived from a consideration of Newton’s law $F = ma$ where $F$ denotes the applied force on a fluid parcel of mass $m$ and $a$ the resultant acceleration. Fluid mechanics in oceanography is based on Newtonian mechanics where we consider conservation of Mass (continuity equation), Conservation of Energy (conservation of heat, heat budgets, mechanical energy, and wave equations) Conservation of Momentum (Navier Stokes Equations) and Conservation of Angular Momentum (conservation of vorticity).

We will use the primitive equations, which are a version of the Navier-Stokes equations that describe hydrodynamic flow on the sphere (the earth) under the assumptions that vertical motion is much smaller than horizontal motion (hydrostatic approximation) and that the fluid layer depth is small compared to the radius of the sphere. Thus, they are a good approximation of global ocean and atmospheric flow and are used in most ocean and atmospheric models. In general, nearly all forms of the primitive equations relate the five variables $u = (u,v,w)$, $T$, $p$ and their evolution over space and time.

2.1 The Vertical Equation

In the vertical direction, the two most important forces are those that result from gravity and changes in pressure. Consider a small cylinder of fluid of mass $m$ as shown in Figure 2.1.

The force downwards due to gravity $g$ is $mg$. The mass of the cylinder also leads to an increase in pressure (force/unit area) from $z_1$ to $z_2 = z_1 + \Delta h$. This increase in pressure leads to a net force upwards given by

$$
V \simeq -\frac{dp}{dz} V
$$

where $V$ is the volume of the cylinder. Now since $m = \rho V$, the acceleration of the cylinder $a = dw/dt$ is equal to the sum of all the forces $F/m$ so that

$$
\frac{dw}{dt} = -\frac{1}{\rho} \frac{dp}{dz} + g
$$

(2.2)

In the absence of vertical motion $w = 0$ and (2.2) reduces to

$$
\frac{dp}{dz} = \rho g
$$

(2.3)

the Hydrostatic Balance, where the gradient of pressure supports the fluid against the force of gravity. Even in the presence of vertical motion the hydrostatic balance is a very good model for long period motions.
Example:
To show this assume that \( w \sim W = 0.01 \text{ms}^{-1} \) and that the period of motions is \( T \sim 1 \text{ day} \) \((8.64 \times 10^4 \text{ seconds})\). \( W \) and \( T \) represent typical values. In this case we scale

\[
\begin{align*}
\frac{dw}{dt} & \sim \frac{W}{T} \sim \frac{0.01}{10^5} \sim 10^{-7} \text{ms}^{-2} \\
& \sim 10^{-7} \text{ms}^{-2}.
\end{align*}
\]

This term may be compared with \( g = 9.8 \text{ms}^{-2} \) and is thus very small. Gravity cannot be balanced by \( dw/dt \) and must be balanced by \(-\frac{1}{\rho} \frac{dp}{dz}\) so that (2.3) is a very good approximation for motions of period a day or more.

To proceed we shall partition the density into two components, \( \rho = \rho_0 + \sigma_T \), where \( \rho_0 = 1000 \text{ kg m}^{-3} \) and as shown in the Figure 2.2, \( \sigma_T \sim 20 \cdots 30 \text { kg m}^{-3} \). Now we shall also define two components of pressure each of which is to balance \( \rho_0 \) and \( \sigma_T \). We define \( p_0 \) by

\[
\frac{dp_0}{dz} = \rho_0 g
\]

and \( p' \) by

\[
\frac{dp'}{dz} = \sigma_T g
\]

so that \( p = p_0 + p' \). The pressure \( p_0 \) is from (2.5) given by

\[
p_0 = \rho_0 g z
\]
and is called the static pressure. It is only related to the constant density $\rho_0 = 1000 \text{ kg m}^{-3}$ and serves to support the bulk of the water mass between $z = 0$ and the depth $z$ (Figure 2.2). The component $p'$ is of more interest since it is related to $\eta$ and $\sigma_T$ and thus to variations in sea level and variations in the distribution of heat (and salt) and, ultimately, to the circulation of the oceans and atmosphere. Since $\sigma_T$ will be a function of $z$ we cannot in general simplify (2.6) further but note that the density field $\sigma_T$ is also in hydrostatic equilibrium.

A major simplification can be made if we can assume that $\sigma_T$ is zero and the density of the ocean is everywhere constant so that $\rho = \rho_0$. In this case (2.6) reduces to

$$\frac{dp'}{dz} = 0 \quad (2.8)$$

and since $p_0$ supports the mass from to the surface to a depth $z$, $p'$ supports the residual mass from $z = 0$ the surface to $z = \eta$ (see the figure) i.e.

$$p' = \rho_0 g \eta \quad (2.9)$$

Note that while $\eta$ may vary with $x, y$ and $t$ it is independent of depth (as is $p'$). An ocean or atmosphere in which density is constant is called barotropic and for ocean shelves, where the

![Figure 2.2: Perturbations in Sealevel](image)
water is well mixed, (Equation 2.9) is a very good approximation. Indeed, measurements of bottom pressure in the ocean (depth \( h \)) can be used to obtain \( p' \) from \( p' = p - \rho_0 g h \) and thus sea level variations \( \eta \). Finally, it is worth noting that by measuring pressure on an instrument that the depth \( z \) of the instrument may be very accurately obtained. Since \( \sigma_T \ll \rho_0 \) we have \( p_0 \gg p' \) so that \( p \simeq p_0 = \rho_0 g z \).

2.1.1 Equations of State

**Oceans**: For the oceans, the density \( \rho \) depends on the temperature \( T \) (°C), the salinity \( S \) and on the pressure \( p \) and can be approximated by a relation known as the equation of state,

\[
\rho = \rho_{00}(1 - \alpha T + \beta S),
\]

where \( \alpha \) and \( \beta \) depend on temperature, salinity and pressure and \( \rho_{00} \) is some constant density. Note that in the relation above, larger temperatures mean the water is less dense while larger values of salinity mean the water is denser.

![Figure 2.3: Typical values of Temperature, Salinity and Density.](image)

**Atmosphere**: For the atmosphere, the density may be written in terms of the temperature \( T \) (expressed in Kelvin; 273 K = 0°C) and to a good approximation, the equation of state is given by

\[
\rho = \frac{p}{RT}
\]

where \( R = 287 \text{ J/(kg K)} \) and \( T \) is given in K. Figure 2.4 shows a mean vertical profile of temperature in the atmosphere, with increasing height above the Earth’s surface. The *Lapse Rate* is the rate at which temperature decreases with height in the atmosphere. This has the opposite sign from the temperature gradient a physicist would use, so be careful. Throughout the bottom 10 – 15 km of the atmosphere the lapse rate is \( \sim 6.5 \text{ K/km} \). This is a typical value, where daytime convection stirs things up.
Over the first 10 km or so is the troposphere which contains 80% of the atmosphere’s mass and nearly all the water vapour. It is characterised by vertical mixing, storms and latent heat release. Jets typically fly above this layer so as to avoid the “weather”.

Above this layer lies the stratosphere which is poorly mixed. The temperature increases with height due to the radiative balances with the sun (discussed later).

From the above, the hydrostatic balance for the atmosphere may be written as

\[ \frac{dp}{dz} = \frac{g \rho}{RT} \]

and from the figure above we take T to be approximately constant and equal to 250 K. In this case we may integrate the above from \( z = 0 \) to some height to obtain

\[ p = p_a \exp\left(\frac{z}{H_c}\right) \]

where \( p_a \) is some surface pressure and \( H_c = RT/g \) is an e-folding scale height equal to about 7 km. That is, over this height the pressure drops by \( 1/e = 1/2.78 \) or about a third. (Recall that we take \( z \) to be negative upwards.)

### 2.2 Buoyancy Effects

Gravity acts on vertical density gradients in the ocean or atmosphere to either stabilise or destabilise the column of fluid. Here we shall quantify how vertical changes in density
can act to stabilise vertical movement of water parcels. Consider an ocean with density dependent only on depth: $\rho = \rho(z)$, with $z$ zero at the surface and increasing with depth. For convenience, let us assume that $\rho$ increases linearly with depth according to:

$$\rho(z) = \rho_0 + kz = \rho_0 + z \frac{d\rho}{dz}$$

(2.10c)

where $k = \frac{d\rho}{dz}$ is a constant. Now if a parcel is moved from $z = z_1$ to a depth $z = z_2$ i.e. it displaces more dense fluid as shown in Figure 2.5.

![Figure 2.5: Oscillation of a parcel of fluid in the ocean.](image)

Archimedes tells us that the force on the parcel will be equal to the mass displaced times gravity $g$ (9.8 m s$^{-2}$) i.e. the acceleration $a$ is

$$a = g(\rho_0 - \rho(z))/\rho_0$$

(2.10d).

If $\frac{d\rho}{dz} > 0$ then $\rho(z) > \rho_0$ so that density at depth is larger than at the surface. In this case $a < 0$ and the acceleration (force/unit mass) is upward. Thus, the tendency is for the parcel to be forced back to its initial position. The negative force acts to push the parcel back to where it came from. It is this restoring force that inhibits mixing in both the atmosphere and the ocean, (hence keeping cold water at depth in the ocean). In addition, the restoring force prevents the ready mixing of greenhouse gases into the ocean. Assuming a linear density gradient (as in Figure 2.5), a movement from $z = 0$ to $z$ will result in a density difference given by

$$\rho' = (\frac{d\rho}{dz})(z - z_o)$$

and the acceleration is

$$\frac{dw}{dt} = -\frac{g \frac{d\rho}{dz}}{\rho_0} (z - z_o)$$
Now with $z$ the position of the parcel, the acceleration is $a = d^2z/dt^2$ ($t$ is time) and putting $\rho(z)$ we get

$$\frac{d^2z}{dt^2} = a = -z \left( \frac{g}{\rho_0} \frac{d\rho}{dz} \right)$$

Let us define

$$N^2 = \frac{g}{\rho_0} \frac{d\rho}{dz}$$

(2.10c)

(we will see why below), so that

$$\frac{d^2z}{dt^2} = -N^2z$$

This is a differential equation for the motion of the fluid parcel. A solution is

$$z = A \sin(Nt)$$

where the amplitude $A$ is unknown. However, we do know that the parcel will move like a spring with frequency $N$, the buoyancy or Brunt-Vaisala frequency.

The period $T$ is given by

$$T = \frac{2\pi}{N}$$

so that if $d\rho/dz$ is large (i.e., the stratification strong), then $N$ is large and $T$ small i.e. fast oscillations in a strongly stratified ocean. If $d\rho/dz$ is small (e.g., in the deep ocean), however, then $N$ is small and $T$ is large. Such waves in the atmosphere can be seen as parallel lines of clouds.

### 2.3 The Horizontal Equations

In the horizontal ($x, y$) plane pressure gradients will also result in forces on fluid parcels. In the following we will consider a barotropic ocean so that these forces may be thought to act over the entire column depth. The forces due to horizontal pressure gradients may be written as

$$-\frac{1}{\rho_0} \frac{dp'}{dx} \quad \text{and} \quad -\frac{1}{\rho_0} \frac{dp'}{dy}$$

(2.11)

and we note that only $p'$ appears since the static pressure $p_0$ is only a function of $z$.

#### 2.3.1 The Coriolis Force

The second force that must be considered is that which arises from the rotation of the Earth. To illustrate this force consider the rotating table as sketched below with two observers $B$, at rest and $A$, spinning with the table. Now if $A$ rolls a ball towards $B$, the observer $B$ will see the ball move in the straight line indicated. Observer $A$ however will say that the ball was deflected to the left. The ‘force’ which does this only appears to the observer in the rotating reference frame and only acts on moving objects.
If $\Omega$ denotes the angular speed of the turntable (radians/sec) then the force is given by $2\Omega U m$ where $U$ and $m$ are the velocity and mass of the ball. For the Earth, the local angular speed varies with latitude. At the equator it is zero, while at the poles it is largest $(2\pi/8 \times 64 \times 10^4 \text{sec})$. A convenient way of expressing this is through the Coriolis parameter

$$f = 2\Omega \sin(\text{latitude})$$

and the acceleration due to the Coriolis force is given by

$$fv \quad \text{and} \quad -fu$$

(2.12)

in the $x$ and $y$ directions respectively. Note $v$ and $u$ are the velocities in the $y$ and $x$ directions. In the northern hemisphere the Coriolis force acts to deflect to the right and $f$ is positive. In the southern hemisphere the reverse holds.

Now collecting the forces we have the following horizontal equations $(a = F/m)$

$$\frac{du}{dt} = -\frac{1}{\rho_0} \frac{dp'}{dx} + fv$$

(2.13a)

$$\frac{dv}{dt} = -\frac{1}{\rho_0} \frac{dp'}{dy} - fu$$

(2.14a)

and in the vertical (2.6):

$$0 = -\frac{dp'}{dz} + \sigma_T g$$

For a barotropic ocean where $\sigma_T = 0$ and $p' = \rho_0 g \eta$ the above simplify to

$$\frac{du}{dt} = -g \frac{d\eta}{dx} + fv$$

(2.13b)

$$\frac{dv}{dt} = -g \frac{d\eta}{dy} - fu$$

(2.14b)
2.4 The Geostrophic Balance

The above equations may be simplified further for motions with periods of 10 days or more. To see this we again perform a scaling analysis on (2.13) and (2.14) and choose a period $T \approx 10$ days $= 8 \cdot 64 \times 10^5$ seconds. (Always work in mks - meters, kilograms and seconds).

With $f \approx 10^{-4}$ sec$^{-1}$ (latitude $\approx 40^\circ$) and assuming that $u \approx v \approx U$, an unknown scale velocity, then

$$\frac{du}{dt} \sim \frac{dv}{dt} \sim \frac{U}{T} \sim U \times 10^{-6} \text{ sec}^{-1},$$

while

$$fv \sim fu \sim fU \sim U \times 10^{-4} \text{ sec}^{-1}.$$ 

Thus, the acceleration terms are a hundred times smaller than the Coriolis terms and negligible for these long period motions. In this case the equations (2.13)–(2.14) reduce to

$$0 = -\frac{1}{\rho_0} \frac{dp'}{dx} + fv \quad (2.15a)$$

$$0 = -\frac{1}{\rho_0} \frac{dp'}{dy} - fu \quad (2.15b)$$

which is known as the geostrophic balance. The deflection by the Coriolis force is now balanced by the force due to the gradients of pressure. As an example consider the atmospheric High–Low system sketched in Figure 2.7. The fluid parcel shown does not rush from High to Low (the pressure gradient force) but instead is balanced by a Coriolis force associated with motion in the $y$ direction.

![Figure 2.7: Geostrophic Balance in the Southern Hemisphere CF=fv and PGF=\(\frac{dp}{\rho dx}\)](image-url)
But how does a geostrophic balance get set up? Consider the atmospheric high-low system (Figure 2.8). We assume that initially \((t = 0)\) there is no motion. However a pressure force exists which will act to accelerate the fluid parcel in the direction shown. As it gets faster, the Coriolis force \(fv\) gets bigger and increasingly deflects the parcel to the left until a geostrophic balance is obtained.

![Figure 2.8: Plan view of adjustment of a water column in the southern hemisphere. The column A is at rest and so only experiences the pressure force as shown. The force accelerates the column but as it moves, the Coriolis force deflects it to the left until a geostrophic balance is achieved at B.](image)

As a second example consider the sea level slope as sketched in Figure 2.9. Since \(\eta\) increases with \(x\), \(\frac{d\eta}{dx} > 0\) and the force \(-g\frac{d\eta}{dx}\) is negative as shown. The geostrophic balance is

\[
0 = -g\frac{d\eta}{dx} + fv
\]

so that \(fv\) is positive and the fluid column must move into the page (the \(y\) direction).

**Exercise:** Calculate the geostrophic velocity in Bass Strait using the data provided by the weather map in Figure 2.10. Note, 1 hectopascal (the units of pressure shown) is equal to 100 Pascals and you should always convert to and work in the units of Pascals, \((\text{Pa})\), meters, \((m)\), kilograms, \((\text{kg})\), seconds \((s)\) as shown in Table 1.9. For the exercise, you can also assume \(f = -10^{-4}\text{s}^{-1}\) and \(\rho_{\text{air}} = 1\ \text{kg m}^{-3}\).
2.5 The Thermal Wind Balance

In the above we have assumed that density $\rho$ is constant. However, small horizontal changes in density can result in large \textbf{vertical} changes in current, particularly near fronts and eddies. For a Geostrophic balance we have from (2.15)
\[ v = \frac{1}{\rho f} \frac{dp}{dx} \quad (2.16a) \]

\[ u = -\frac{1}{\rho f} \frac{dp}{dy} \quad (2.16b) \]

and the hydrostatic balance

\[ \frac{dp}{dz} = \rho g \]

Now differentiating (2.16a) and (2.16b) with respect to \( z \) (depth) we get

\[ \frac{dv}{dz} = \frac{1}{\rho_0 f} \frac{d}{dz} \left( \frac{dp}{dx} \right) = \frac{1}{\rho_0 f} \frac{d}{dx} \left( \frac{dp}{dz} \right) \quad (2.17a) \]

\[ \frac{du}{dz} = \frac{1}{\rho_0 f} \frac{d}{dy} \left( \frac{dp}{dz} \right) \quad (2.17b) \]

where since \( \rho \) is very nearly constant, the density in the denominator of (2.7) is taken as the constant \( \rho_0 \).

Now substituting for \( dp/dz \) we get

\[ \frac{dv}{dz} = \frac{g}{\rho_0 f} \frac{dp}{dx} \quad (2.18a) \]

\[ \frac{du}{dz} = -\frac{g}{\rho_0 f} \frac{dp}{dy} \quad (2.18b) \]

the ‘thermal wind’ balance. The balance (2.18) may again be thought of in terms of forces. Consider the sloping density surfaces shown below. Due to the weight of the water, the relatively denser water on the right leads to a pressure force (PF) as shown and this is balanced by a Coriolis force.
The motion is into the page (northern hemisphere) and gets larger with depth \((dv/dz > 0)\) since the horizontal pressure gradient also increases with depth (more relatively dense water above).

*An example: a cold core eddy.*

Suppose we wish to compute the change in geostrophic velocity with height for the density field of the eddy shown below. Note that at a depth of 500 m, the centre is denser than the surrounding fluid.

![Figure 2.11](image_url)

**Figure 2.11:**

![Figure 2.12](image_url)

**Figure 2.12:** Distribution of potential density \(\sigma_\theta(\text{kg m}^{-3})\) for a CTD section through Eddy Bob (*Olson*, 1980).
Now let us calculate \( \Delta v = \frac{\partial v}{\partial z} \Delta z \) at \( x = 50 \) km and at a depth of 500 m. Taking \( y \) into the page, we can estimate

\[
\frac{dv}{dz} = \frac{g}{\rho_0 f} \frac{d\rho}{dx} \simeq \frac{g}{\rho_0 f} \Delta \rho
\]

where

\[
\Delta \rho \simeq \rho(50 \text{ km}) - \rho(0) = 1027.0 - 1027.6 = -0.6 \text{ kg m}^{-3}
\]

and

\[
\Delta x = (50 - 0)\text{km} = 50 \times 10^3 \text{m}.
\]

With \( g = 10, \rho_0 \simeq 1027, f = 10^{-4}\text{s}^{-1} \), we get

\[
\frac{dv}{dz} = \frac{10}{1027 \times 10^{-4}} \frac{(-0.6)}{50 \times 10^3} = -12 \times 10^{-3} \text{s}^{-1}.
\]

The change in \( v \) from \( z = 500 \) to \( z = 750 \) m is then

\[
\Delta v \simeq \frac{dv}{dz} \Delta z = -1.2 \times 10^{-3} \times 250 = -0.3 \text{ ms}^{-1}, \text{i.e. 30 cm s}^{-1}
\]

As we go to great depth \( \frac{d\rho}{dx} \simeq 0 \), and for large \( x \) (i.e. far from the eddy) \( \frac{d\rho}{dx} = 0 \). Thus \( \frac{dv}{dz} = 0 \) in each of these cases. We can thus infer the velocities. A cold core eddy is like a low pressure system, (or cyclone) and follows the right hand rule in the Northern hemisphere.
Figure 2.13: Velocity and Stratification through a cold core eddy (left) and a warm-core eddy (right) (Olson, 1991).

2.5.1 An Example: an atmospheric front

For the atmosphere we have

\[ f \frac{\partial v}{\partial z} = \frac{g}{\rho} \frac{\partial \rho}{\partial x}, \quad f \frac{\partial u}{\partial z} = -\frac{g}{\rho} \frac{\partial \rho}{\partial y} \]

and \( \rho = p/RT \). It turns out that changes of \( p \) with \( x \) and \( y \) can be neglected compared to changes in \( T \) so that the above become

\[ f \frac{\partial v}{\partial z} = -\frac{g}{T} \frac{\partial T}{\partial x}, \quad f \frac{\partial u}{\partial z} = -\frac{g}{T} \frac{\partial T}{\partial y}. \]  

(2.19)

Now consider a cold front that is being advected at a speed \( U_0 \) towards us (Figure 2.14):

In region (1) the hot air is well mixed and \( \frac{\partial T}{\partial x} = 0 \) so that \( v \) is zero or constant. In region (2), the front exists and since \( \frac{\partial T}{\partial x} \) is positive, \( f \frac{\partial v}{\partial z} \) is negative. In the southern hemisphere \( f \) is negative and so \( \frac{\partial v}{\partial z} \) is positive and \( v \) increases from zero (above the front) to some positive
value below the front. That is the temperature suddenly drops and the wind veers abruptly to the right if we are facing the front.

How big is the wind? Suppose $T$ changes by $\Delta T = 10^\circ C$ over $\Delta x = 100$ km then $\frac{\partial T}{\partial x} \sim \frac{\Delta T}{\Delta x} \sim 10^{-4}$ Km$^{-1}$. With $g = 10$ m s$^{-2}$, $f = -10^{-4}$ s$^{-1}$ and $T \simeq 300$K we get

$$\frac{\partial v}{\partial z} = -\frac{g}{fT} \frac{\partial T}{\partial x} \simeq 0.03$s$^{-1}.$$ 

The change $\Delta v$ over some height $\Delta z$ is then

$$\Delta v \simeq \frac{\partial v}{\partial z} \Delta z$$

and if we take $\Delta z$ to be 2000 m we get $\Delta v \simeq 60$ ms$^{-1}$ or 200km/hr!

### 2.6 Taylor Sheets, Columns and Blocking

The thermal wind relations are

$$f \frac{\partial v}{\partial z} = \frac{g}{\rho_0} \frac{\partial \rho}{\partial x},$$

$$f \frac{\partial u}{\partial z} = -\frac{g}{\rho_0} \frac{\partial \rho}{\partial y},$$

so that if the ocean/atmosphere is homogeneous (well mixed) and $\rho = \rho_0$ a constant then

$$\frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} = 0, \quad (2.20)$$
and the horizontal velocities \((u, v)\) do not vary with height or depth. This ‘vertical rigidity’ is a fundamental property of a rotating homogeneous fluid.

The nature of the vertical velocity can also be determined. Since the motion is also geostrophic we have

\[
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( \frac{1}{\rho_0 f} \frac{\partial p}{\partial y} \right) = -\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y} \left( \frac{-1}{\rho_0 f} \frac{\partial p}{\partial x} \right)
\]

so that \(u_x + v_y = 0\). Now it can be shown that the velocity field also satisfies the condition

\[ u_x + v_y + w_z = 0. \]

In this case, if \(u_x + v_y = 0\) then we must have

\[
\frac{\partial w}{\partial z} = 0 \tag{2.21}
\]

as a corollary of (2.20) above, and \(w\) also is independent of \(z\).

2.6.1 Experiment: Taylor Curtains

![Taylor Curtains](Image)

Figure 2.15: Taylor Curtains (Cushman-Roisin and Beckers, 2007)

2.6.2 Taylor Columns:

An oceanographic example of a Taylor Column is the example of flow post a seamount (Figure 2.16). Since \((u, v, w)\) cannot vary with \(z\), the flow is around the seamount while above it there is no motion: Implications for marine ecology.
2.6.3 Blocking

The onshore sea breeze below will be blocked by the presence of the mountain range just as in the case of the Taylor column: Implications for pollution in Sydney’s west.

Figure 2.17: Blocking of the onshore seabreeze in Sydney’s west.
2.7 The effects of Friction

The presence of turbulence acts to increase the frictional drag of the oceans and atmosphere, notably near coastal boundaries and the air-sea interface. For motions of period 3 days or longer a simple model for the frictional force due to the sea floor (or land) is given by

\[ -ru/h \quad \text{and} \quad -rv/h \]

in the \( x \) and \( y \) directions where \( h \) is the ocean depth and \( r \) a coefficient given by

\[ r = C_D v_* \]

\( C_D \simeq 2 \times 10^{-3} \) is a non–dimensional drag coefficient and \( v_* \) denotes a typical turbulent or tidal velocity. The equations of motion (2.13) and (2.14) then become

\[
\frac{du}{dt} = -g \frac{d\eta}{dx} + f v - ru/h \\
\frac{dv}{dt} = -g \frac{d\eta}{dy} - f u - rv/h. \tag{2.22, 2.23}
\]

The effects of friction are readily seen if we consider the simpler balance

\[
\frac{du}{dt} = -ru/h.
\]

A solution is

\[ u = u_0 e^{-rt/h} \]

so that at \( t = 0 \) \( u = u_0 \) while as \( t \) becomes large, \( u \) becomes small. Clearly \((h/r)\) plays the role of an \( e \)–folding time scale of frictional spin down. For \( r = 5 \times 10^{-5} \text{m sec}^{-1} \) and \( h = 70 \text{m} \) (Bass Strait) \( h/r = 16.2 \text{ days} \). Friction thus acts to damp out motion.

2.8 Conservation of Mass and Compressibility in the Ocean and Atmosphere

Conservation of mass is an important constraint on fluid motion. If we consider a fixed fluid volume, the mass of the fluid occupying the volume may change with time if the density changes, however mass continuity tells us that this can only occur if there is flux of mass into or out of the volume.

\[
\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \tag{2.24}
\]

The definition of incompressible flow is that it is non-divergent, however for any real fluid, it is never exactly obeyed. For incompressible flow such as liquid in the ocean we can
consider a simplified approximation to the continuity equation. For the barotropic ocean with variations in sea level it may be written as:

\[ \frac{d\eta}{dt} + \frac{d(hu)}{dx} + \frac{d(hv)}{dy} = 0 \]  

(2.25)

and expresses the fact that as sea level changes, water must flow in or out in compensation (Figure 2.18). For \( v = 0 \) and \( h = \text{constant} \) (2.22) becomes

\[ \frac{d\eta}{dt} + h \frac{du}{dx} = 0. \]

In comparison, a compressible fluid such as air is nowhere close to being non-divergent, this is because the density changes drastically as the fluid parcels expand and contract. This is very inconvenient in the analysis of atmospheric dynamics. A simplification exists if the hydrostatic approximation is valid, and if we adopt pressure co-ordinates. In the pressure coordinate version of the continuity equation there is no term representing the rate of change of density it is simply:

\[ \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dp} = 0 \]  

(2.26)

The simplicity of this equation is one reason why pressure coordinates are favoured in meteorology.
3 Unforced Motions: Waves

The goal of this chapter is two fold, 1. to introduce the concepts of waves, and some simple aspects of wave theory, and 2. to relate this to examples in the ocean and the atmosphere. We will consider unforced motions that satisfy the equations of motion (2.13)–(2.14) and (2.22). We will begin with surface gravity waves where the effects of the Earth’s rotation are unimportant and then consider waves for which this is not the case. This will lead to the concept of the potential vorticity (or spin) of a fluid column which is vital in understanding large scale ocean circulation.

3.1 Long Surface Gravity Waves

Such waves are simply those seen at the beach in the region before the surf zone. To model these waves we will only consider motion in the \((x, z)\) plane as shown in Figure 3.1.

In addition we will need to assume that the wavelength \(\lambda\) is much larger than the water depth \(h\) (else the hydrostatic approximation will be invalid). The effects of friction and of the Earth’s rotation may also be ignored. (Show this using a scaling argument with \(T \sim 10\) seconds).

Now since the coastal ocean can often be well mixed we assume a barotropic ocean of constant depth \(h\). The equations (2.13) and (2.22) are then

\[
\frac{du}{dt} = -g \frac{d\eta}{dx}
\]  

(3.1)
\[ \frac{d\eta}{dt} + h \frac{du}{dx} = 0 \] (3.2)

and by differentiating Eq 3.1 with respect to \( x \) and Eq 3.2 with respect to \( t \) we obtain the wave equation.

\[ \frac{d^2\eta}{dt^2} - gh \frac{d^2\eta}{dx^2} = 0 \] (3.3)

Let us consider a plane wave solution of the form

\[ \eta = A \cos(kx - \omega t) \] (3.4)

where \( \omega = \frac{2\pi}{T} \) is the frequency and \( k = \frac{2\pi}{\lambda} \) is the wavenumber.

The wave period \( T \) is the time it takes two successive wave crests or troughs to pass a fixed point. The wave length, \( \lambda \) is the distance between two successive waves crests or troughs at a fixed time. The wave number, \( k \) represents the number of crests in the \( x \) direction.

Frequency and period are distinctly different, yet related, quantities. Frequency refers to how often something happens; period refers to the time it takes something to happen. Frequency is a rate quantity; period is a time quantity. Frequency is the cycles/second; period is the seconds/cycle.

### 3.1.1 Dispersion relation

Wave frequency \( \omega \) is related to wave number \( k \) by the dispersion relation

\[ \omega^2 = g k \tanh(kh) \]

Approximations can be made in both shallow and deep water. These will be discussed further in Section 3.3. Here we will investigate the shallow water wave approximation, which is valid if the water depth is much less than a wavelength. In this case \( h << \lambda, \; kh << 1 \) and \( \tanh(kh) = kh \).

Substituting Eq 3.4 into Eq 3.3 we get

\[ -\omega^2 A \cos(kx - \omega t) + h g k^2 A \cos(kx - \omega t) = 0 \]

Cancelling the term \( A \cos(kx - \omega t) \) results in the shallow water dispersion relation:

\[ \frac{\omega^2}{k^2} = hg \]

or

\[ c = \frac{\omega}{k} = \sqrt{gh} \] (3.5)

The quantity \( c \equiv \omega/k \) is called the phase speed and is the rate at which wave crests (or troughs) pass a fixed point. This may be seen from Eq 3.4 where for \( \eta = A \) (a crest) we must have \( kx - \omega t = 0 \) (say) so that we must change \( x \) by the amount

\[ x = \frac{\omega t}{k} \]

to keep up with the crest (Figure 3.1).
Now returning to Eq 3.5, the wave is only a solution if \( c = \sqrt{gh} \), so that if \( h = 2 \) m then \( c = 4.4 \) ms\(^{-1}\). Given we know the wave period, e.g \( T = 3 \) sec, then since \( c = \omega/k = \lambda/T \) we have \( \lambda = cT \) so that \( \lambda = 13.3 \) m.

### 3.2 Wave Refraction, Diffraction and Shoaling

#### 3.2.1 Refraction

If you look out from any headland you will see that wave crests tend to become parallel to the shore as the wave moves inshore.
Exercise: A wave has a frequency \( \omega = 2\pi/4s^{-1} \) in 100 m of water. Calculate \( c \) and \( \lambda \). Calculate \( c \) and \( \lambda \) after the wave has reached a depth of 10 m of water. Explain what has happened.

Now consider two parts \( I \) and \( J \) of a wave crest. Offshore, the crests move at a speed \( c_I = c_J = \sqrt{gh} \approx \sqrt{10 \times 50} \text{ ms}^{-1} \). As the wave moves further in, the crest \( J \) is in deeper water so that

\[
c_J = \sqrt{gh_J} > c_I = \sqrt{gh_I}
\]

since \( h_J > h_I \). Thus, the wave crest at \( J \) moves faster and the wave tends to become more parallel with the shoreline.

### 3.2.2 Shoaling

Shoaling is the term for the changes in wave characteristics that occur when a wave reaches shallow water. As well as slowing, waves steepen as the depth decreases. This observation may be explained using the following formulae for the flux of energy \( \Gamma \):

\[
\Gamma = cE \quad (3.6)
\]

where \( c \) is the wave phase speed and

\[
E = \frac{1}{2}\rho_0 g A^2 \quad (3.7)
\]

is the total kinetic and potential energy of the wave (due to motion and sea-level variations) for

\[
\eta = A \cos(kx - \omega t).
\]

Now consider a profile of the beach shown in Figure 3.3. In the absence of friction and wave breaking the energy flux at \( I \) is equal to that at \( J \) so that

\[
\Gamma_I = c_I E_I = \Gamma_J = c_J E_J \quad (3.8)
\]

and

\[
E_J = (c_I / c_J)E_I \quad (3.9)
\]

From Eq 3.7 we then have

\[
A_J^2 = (h_I / h_J)^{\frac{3}{2}} A_I^2
\]

so that with \( h_J < h_I \), the amplitude of the wave \( A_J \) is larger than \( A_I \), i.e. the wave steepens.

In summary the decreasing depth causes: (a) An increase in wave height - The conservation of energy results in more energy forced into a smaller area. Since wave energy is proportional to wave height squared, this increases wave height as it propagates toward shore even though some of the energy is dissipated by bottom friction.
(b) A decrease in wave speed - Remember that waves in shallow water have speeds that are dependent on the square root of water depth. As the depth decreases, so too will the wave speed.

Refraction is the change in direction of a wave due to a change in its speed. This is most commonly seen when a wave passes from one medium to another, or in the case of the ocean, moves from deep water into shallow water. Refraction of ocean waves generally occurs as they approach the shore. This process unevenly distributes wave energy along the shoreline, and results in erosion at headlands and deposition on beaches.

Diffraction refers to the various phenomena associated with wave propagation, such as the bending, spreading and the interference of waves passing by an object. In the ocean wave diffraction results from wave energy being transferred around or away from barriers impeding its forward motion. e.g. waves move past barriers into harbours because their energy moves laterally along the crest of the wave, and the wave behind the barrier goes out in all directions (Thurman and Burton, 2001)

### 3.2.3 Wave Breaking

As the wave moves into shallower water, shoaling affects the wave form by slowing its base while having less effect on the crest. At some point, the crest of the wave is moving too fast for the bottom of the wave form to keep up. The wave then becomes unstable and breaks. For a wave of the form

$$\eta = A \cos(kx - \omega t)$$
observations show that steepening will occur and the wave will break if the amplitude gets sufficiently large and

\[ A \gtrsim \lambda/12. \]

Alternatively, if the water gets too shallow, the wave will break if

\[ A \gtrsim 0.8h. \]

In meteorology, gravity waves are said to break when the wave produces regions where the potential temperature decreases with height, leading to energy dissipation through convective instability; likewise Rossby waves are said to break when the potential vorticity gradient is overturned.

### 3.3 Shallow and Deep Water Waves

The waves described above are restricted to the case where \( \lambda \gg h, \) or \( kh \ll 1. \) More generally, it can be shown that

\[ c = \frac{\omega}{k} = \left[ \frac{g}{k} \tanh(kh) \right]^{\frac{1}{2}} \quad (3.10) \]

and if \( \lambda \gg h(kh \ll 1) \) this reduces to shallow water waves above where \( c = \frac{\omega}{k} = \sqrt{gh}. \) Where \( \lambda < 2h \) we have deep water waves and Eq 3.10 becomes

\[ c = \frac{\omega}{k} = (g/k)^{\frac{1}{2}} = (g\lambda/2\pi)^{\frac{1}{2}} \]
3.4 Resonance

Tides are long gravity waves driven by the gravitational attraction of the moon, sun and Earth. The amplitude of the tide may be greatly enhanced by resonance with bays of particular sizes. Consider a bay of length $L$:

Now suppose the tides drive a sea level signal at $x = L$ given by

$$\eta_{\text{Tide}} = \eta_0 \cos \omega t,$$

where $\omega = 2\pi/(\text{tidal period})$ and $\eta_0$ is constant. Now to illustrate resonance, assume a solution to Eq 3.3, the wave equation,

$$\eta = A \cos kx \cos \omega t,$$  \hspace{1cm} (3.11)

where again $c = \frac{\omega}{k} = \sqrt{gh}$. At $x = 0$, $u_t = -g\eta_x = 0$ since $u = 0$. Equation 3.11 satisfies this, since $\eta_x \propto \sin kx = 0$. Equation 3.11 is also a solution to the wave equation if $c = \frac{\omega}{k} = \sqrt{gh}$.

At $x = L$, we match sea-level of the solution to $\eta_{\text{Tide}}$

$$\eta = A \cos kL \cos \omega t = \eta_0 \cos \omega t$$

so that $A$ is given by

$$A = \frac{\eta_0}{\cos kL}.$$  

Now $\cos kL$ will be zero if $kL = \frac{\pi}{2} (+ 2n\pi)$. With $c = \frac{\omega}{k}$, $k = \frac{\omega}{c}$ the condition $kL = \frac{\pi}{2}$ will be met if

$$L = \frac{\pi}{2k} = \frac{\pi c}{2\omega} = \frac{cT}{4}.$$
If the length of the bay is such that

\[ L = \frac{cT}{4} = \frac{T}{4}\sqrt{gh} \]

then \( \cos kL \simeq 0 \) and \( A = \frac{\eta}{\cos(kL)} \) becomes infinite. In practice, frictional effects due to bottom interaction prevent the resonance going to infinity. This is called a quarter wave resonator and sea level is as shown in Figure 3.4.

Examples: Bay, of Fundy (Canada) \( A \sim 10 \) m!. Broad sound (Qld) \( A \sim 3 \) m. Another example is Spencer Gulf. Show it is a 1/4 wave resonator for the M2 and S2 tides.

### 3.5 Potential and Relative Vorticity

In the absence of forcing, frictional effects and density changes, a fundamental quantity of importance is related to the “spin” or vorticity of a fluid column which requires that the quantity

\[ Q = \frac{\zeta + f}{h + \eta} \]

be conserved following a fluid column.

As we will see the fact that \( Q \) is conserved (unchanged) following a fluid column will permit us to understand ocean fronts and large-scale ocean circulation. Now to understand Eq 3.12 let us consider the Coriolis parameter \( f \). It may be regarded as the spin of a fluid column as seen from outer space in the absence of any other motion. Indeed, at the poles \(|f| = 2\Omega\), where \( \Omega = 2\pi \) radians/day.

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The new additional term in Eq 3.12 $\zeta$ is called the relative vorticity and is related to the spin we register here on Earth and also related to the shear of the velocity field:

$$\zeta = \frac{dv}{dx} - \frac{du}{dy}$$

As shown below, a cork near a bathtub wall will spin since at the wall the velocity $v = 0$ (friction) while further out $v \neq 0$. Since $dv/dx > 0$, $\zeta > 0$ and the spin is positive: we use a right hand rule as shown.

![Figure 3.4](image)

Figure 3.4: Left: rotation of the earth relative to the north pole. Right: Spin of a cork due to velocity shear.

Now the total spin of a fluid column will consist of both $\zeta + f$ as observed from space. The length $h + \eta$ of the fluid column becomes important when we note that the potential vorticity or total spin is conserved (unchanged) following a fluid column, i.e.

$$\frac{d}{dt} \left( \frac{\zeta + f}{h + \eta} \right) = 0$$  \hspace{1cm} (3.13)

This result may be understood by considering the skater who with arms outstretched has a spin $f$ and $Q = f/h$. As she brings her arms in her effective height increases. Now since her initial and final potential vorticity remain the same

$$Q_i = \frac{f}{h_i} = Q_f = \frac{f + \zeta}{h_f}$$

and $h_f > h_i$ we have an increase in her relative vorticity:

$$\frac{\zeta_f}{f} = \frac{h_f}{h_i} - 1,$$  \hspace{1cm} (3.14)

i.e. she spins faster, (right hand rule). The same is also true for the fluid column sketched in
Figure 3.5 and cyclonic vorticity is acquired ($\zeta/f > 0$) for vortex stretching and anticyclonic vorticity ($\zeta/f < 0$) for vortex squashing: the column of fluid might be regarded as a vortex.

**An Example: Geostrophic Adjustment**

Suppose that water has been piled up against the coast as shown in Figure 3.6 but that there is no motion. Now to infer the final sea-level and velocity field we expect that the water will move away from the coast (due to the pressure gradient), but that ultimately a sea-level slope may exist that is in geostrophic balance

$$fv = g \frac{d\eta}{dx}$$

In addition we may consider the conservation of potential vorticity

$$Q = \frac{(\zeta + f)}{h_{(total)}}$$

of three columns (a), (b) and (c) as shown during the adjustment process and also note that

$$\zeta = \frac{dv}{dx}$$
At column (c) nothing happens since the depth does not change, \( d\eta/dx = 0 \) and so \( v = 0 \).

Figure 3.6: Left: Piling of water against the coast, Right: Geostrophic adjustment of three columns of water and subsequent change in sea level.

The velocity \( u \) in the \( x \) direction will also everywhere zero since we assume there is no pressure gradient in the \( y \) direction. For the other columns (3.12) can be rewritten as

\[
\frac{\zeta_f}{f} = \frac{h_f}{h_i} - 1,
\]

where the subscripts \( i \) and \( f \) refer to the initial and final states. For column (b) stretching occurs so \( \zeta_f/f > 0 \) and the spin will be as for the right hand. In this case we then have \( dv/dx > 0 \) so that \( v \) must be increasing. For the column (a), squashing occurs and \( \zeta_f/f < 0 \), left hand spin, and \( dv/dx < 0 \) so that \( v \) is decreasing. Putting this information together we get the following picture of the velocity field (as seen from above):

Figure 3.7: Plan view of velocity vectors \( v \)

An exercise:

A fluid column or eddy at the north pole is observed by a spy satellite to be rotating at a rate of \( 2(2\pi + 0.12\pi) \) radians per day. (1 day = \( 24 \times 3600 \) seconds). Given \( f = 2(2\pi/86400) \text{ s}^{-1} \) calculate \( \zeta \). (a) If the fluid column of depth \( h_I = 4000 \text{ m} \) passes over a ridge (\( h_f = 3500 \text{ m} \)) describe the increase in \( \zeta \) using the concept of conservation of potential vorticity. (b) If \( \zeta = dv/dx \) is constant within the eddy of radius 100km, calculate \( v \) at \( x = 100 \text{ km} \) both before and after it passes over the ridge (assume \( v = 0 \) at \( x = 0 \), the center of the
(c) If \( v \) is in geostrophic balance calculate the sea-level field of the eddy both before it crosses the ridge. Assume \( \zeta = 0 \) at the edge of the eddy.
4 Wind Forced Motions

The wind blowing on an ocean produces a force/unit area called a wind stress $\tau$. The surface wind stress can be related to the wind velocity through a relationship known as the 'bulk formula'.

$$(\tau_x, \tau_y) = \rho_{\text{air}} c_D u_{10}(u_a, v_a)$$

(4.1)

where $\tau_x$ and $\tau_y$ are respectively the zonal ($x$ direction) and meridional ($y$ direction) components of wind stress, $c_D$ is a bulk transfer coefficient for momentum (typically $c_D = 1.5 \times 10^{-3}$), $\rho_{\text{air}}$ is the density of air at the sea surface ($\sim 1.3$ kg m$^{-3}$) and $u_{10}$ is the speed of the wind at a height of 10 m above the ocean.

4.1 Ekman Layer

The effects of a wind stress on the ocean surface are transmitted down through the water column by the action of turbulent eddies that are themselves generated by the wind, breaking waves and boundary shear stresses. The depth to which the effects of wind are felt is called the Ekman layer thickness

$$H_E = (2K/|f|)^{\frac{1}{2}}$$

(4.2)

where $K$ is the eddy diffusivity and $K \simeq W L$ where $W$ and $L$ denote a characteristic eddy velocity and size. Typically $K \simeq 2 \times 10^{-2}$ m$^2$ s$^{-1}$ so that with $|f| = 10^{-4}$ s$^{-1}$, $H_E \simeq 20$ m. However in regions of high wind-driven turbulence, $K$ can be up to $0.5$ m$^2$ s$^{-1}$, so that $H_E$ can reach $\simeq 100$ m.

Now to understand the effects of wind consider the steady equations of motion for a barotropic ocean of depth $h$. In the absence of wind and if the bottom stress is weak, these equations of motion would be the geostrophic equations plus the effects of wind forcing, which then enter as an extra frictional term, that is

$$-fv = -g \frac{d\eta}{dx} + \frac{\tau_x}{\rho h}$$

(4.3)

$$fu = -g \frac{d\eta}{dy} + \frac{\tau_y}{\rho h}$$

(4.4)

where $\tau_x$ and $\tau_y$ are the $x$– and $y$– direction components of the wind stress. Now for simplicity, let's split the velocity field $u$ and $v$ into geostrophic and wind-driven components; that is:

$$u = u_g + u_e$$

(4.5)

$$v = v_g + v_e$$

(4.6)
where \((u_g, v_g)\) denote geostrophic velocities such that there is a balance between the pressure gradient and the Coriolis force:

\[
-fv_g = -g \frac{dq}{dx} \quad (4.7)
\]
\[
 fu_g = -g \frac{dq}{dy} \quad (4.8)
\]

Hence the remainder of velocity \((u_e, v_e)\) must balance the force of the wind:

\[
-fv_e = \frac{\tau_x}{\rho h} \quad (4.9)
\]
\[
 fu_e = \frac{\tau_y}{\rho h} \quad (4.10)
\]

The velocities \((u_e, v_e)\) are called the (depth-averaged) **Ekman velocities** and while driven by the wind stress \((\tau_x, \tau_y)\), the velocities \((u_e, v_e)\) are at right angles to it.

### 4.2 The Ekman Velocity

To understand the effects of the wind let us assume it blows in the \(y\) direction only with a force per unit area (a stress) denoted by \(\tau_y\). Assuming there are no sea level gradients, the equations of motion, averaged over the Ekman layer depth may be written as

\[
\frac{dU_E}{dt} - fV_E = 0 \quad (4.11)
\]
\[
\frac{dV_E}{dt} + fU_E = \frac{\tau_y}{\rho H_E} \quad (4.12)
\]

The horizontal velocity averaged over the Ekman layer is denoted by \((U_E, V_E)\). Initially there is no motion and the wind accelerates the column shown in Figure 4.1 in the \(y\)-direction according to \(dV_E/dt = \tau_y/\rho H_E\).

As the column moves faster, the Coriolis force \(fV_E\) gets larger and the column is deflected to the right (Northern Hemisphere): \(dU_E/dt = fV_E\). This continues until \(V_E\) vanishes and a steady state is reached

\[
 fU_E = \frac{\tau_y}{\rho H_E} \quad (4.13)
\]

and the Coriolis force and wind stress balance. The column moves at right angles to the wind. Hence the force balance on the fluid column is in this case between the Coriolis force and the force of the wind.
Figure 4.1: Plan view of the motion of a column of water subject to an initial acceleration by the wind.

An example:
Calculate the Ekman velocity $U_E$ given a wind stress $\tau_y = 0.1$ Pa (0.1 Nm$^{-2}$) Assume an Ekman layer depth of $H_E = 20$ m, and $f = 10^{-4}$ s$^{-1}$ (Equation 4.13). Compare this to the conditions off northern California (Figure 4.6).

4.3 Depth Averaged Ekman Layer

Now to see what happens with depth we note that the wind only directly affects motions over the depth $H_e$, not the entire depth $h$. Thus the velocity $u_e,v_e$ only occurs in the Ekman layer and its value here must be $(h/H_e)$ times the full depth-averaged velocity $u_e,v_e$. This is called the Ekman layer velocity and is given by

\[
U_e = \frac{h}{H_e} u_e = \frac{\tau_y}{\rho H_e f} \tag{4.14}
\]
\[
V_e = \frac{h}{H_e} v_e = -\frac{\tau_x}{\rho H_e f} \tag{4.15}
\]

since $u_e$ and $v_e$ represent velocity averaged over depth $h$. As shown in Figure 4.2, a north-south wind stress drives a velocity $U_e$ in the east-west direction - that is, at right angles to the wind. Checking the signs in Equations 4.14 and 4.15 we can see that flow is to the right in the northern hemisphere, and to the left in the southern hemisphere.
Figure 4.2: Side view of the movement of a column of water in the Ekman Layer in the Northern Hemisphere. Note: There is no motion below the Ekman layer as there is no slope in sea level, hence the geostrophic velocities are zero.

An example

a) Find the Ekman layer depth $H_e$ and the Ekman velocities $u_e$ and $v_e$ when a wind of strength 0.5 Pascals (0.5 Newtons/m$^2$) blows in a north-easterly direction in the mid-latitudes of the northern hemisphere. You can take the vertical diffusivity to be $K = 0.1$ m$^2$/sec and $f = 10^{-4}$ sec$^{-1}$.

b) What is the Ekman layer velocity $U_e, V_e$ if the depth of the ocean $h = 1000$ m?

c) Sketch the resultant flow.

4.4 Storm surges and Downwelling and Upwelling

4.4.1 Storm Surge

Let us now consider the effects of an alongshore wind stress on coastal circulation. Our model is shown in Figure 4.3.

The effect of the coast is to block the onshore Ekman flux resulting in the raising of sea-level. Since $\eta$ now changes with $x$, an alongshore geostrophic velocity

$\nu_g = \frac{g}{f} \frac{d\eta}{dx} < 0$

must exist to balance the force due to $d\eta/dx$.

Thus wind forcing results in the transport of water towards the coast in a thin layer $H_E$, which can in turn raise sea-level and drive an alongshore velocity.
4.4.2 Downwelling

Where the ocean is stratified (density increases with depth), the onshore Ekman flux also pushes water down along the continental shelf leading to mixing and the growth of thick bottom boundary layers. The thermal wind shear near the shelf opposes, the alongshore current \( v_g = (g/f)dy/dx \) and an undercurrent flowing in the opposite direction can result.

**Exercise:** Use the thermal wind relation (2.18) and equation of state (2.10a) to show this is so.
4.4.3 Upwelling

Now consider the two–layer coastal model in Figure 4.5. With the wind stress directed into the page the Ekman flux is directed off–shore so that the coastal sea–level must drop. However, we might expect the interface to rise as well and water (and nutrients) to be drawn from the deep ocean as shown.

Figure 4.5: Schematic diagram of wind driven upwelling in the Northern Hemisphere.

Ekman layer depths vary with wind forcing (strength and duration). An example is found in the Northern Californian upwelling region during the WEST study (Largier et al., 2006; Roughan et al., 2006). Figure 4.6 from (Dever et al., 2006) shows data from a mooring in 90 m of water, showing along and across shelf wind (ms$^{-1}$), temperature ($^\circ$C) and cross-shore and along shore velocities throughout the water column. The coldest (warmest) temperatures correspond with the periods of strong (weak) alongshore wind forcing, and offshore (onshore) transport in the surface waters.

While coastal upwelling occurs in only about 1% of the oceans area, the increased biological productivity accounts for about 50% of the worlds fish catch. Figure 4.7 shows the distribution of winds over the ocean, Note that the coast of Peru, west Africa and California are subject to strong upwelling favourable winds.
4.5 Ekman Pumping

In the above we have implicitly assumed that the wind stress $\tau_y$ and thus Ekman flux $U_E$ are constant. What if $\tau_y$ changes in space (e.g. with $x$). In the schematic in Figure 4.8 it is apparent that the flux $U_E$ will converge at the origin leading to a flux of water $w_E$ out of the Ekman layer. The velocity $w_E$ is called the Ekman pumping velocity and given by

$$w_E(z = H_E) = -H_E \frac{dU_E}{dx}$$ (4.16)
Figure 4.7: Map showing the annual mean distribution of winds around the world. Which regions are upwelling favourable?

Figure 4.8: Ekman Pumping, resulting from a curl (change) in the wind stress.
or

\[ w_E = -\frac{d}{dx}(\tau_y/\rho f) \]  \hspace{1cm} (4.17)

It corresponds to a velocity of fluid leaving or entering the base of the Ekman layer so as to compensate the horizontal variations in the horizontal flow due to the Ekman flux \( U_E \).

An example: a hurricane
Consider the single layer ocean in Figure 4.9 that is subject to the wind stress field shown (a hurricane).

Figure 4.9: Schematic Diagram of the sea surface response to a Hurricane . Note that the Ekman layer thins leading to vortex stretching of the lower layer \( (\zeta/f \geq 0) \).

For the northern hemisphere \( U_E \) is directed to the right of the wind \( \tau \) so that there is a loss of fluid in the Ekman layer. However, this fluid is replaced by the upward Ekman pumping velocity \( w_E \). Note that not all the fluid is replaced and we might expect sea-level to dip in the centre as shown.
Figure 4.10: Two sections showing isopycnals as observed several days after the passage of a Hurricane across the Gulf of Mexico. The offshore Ekman transport and resultant Ekman upwelling has raised the isopycnals by more than 60 m near the hurricane center.

Exercise
Use the thermal wind relations to estimate the change in velocity over a depth of $80 - 100$ m at the position indicated by the arrow for section C-C'.

4.6 Circulation along the Equator

There is a complicated current structure near the equator which is associated with the wind driven circulation in each hemisphere. The North and South Equatorial Currents (NEC and SEC) are westward currents that are associated with the wind gyres with speeds of $25 - 30$ cm$^{-1}$ and $50 - 65$ cm$^{-1}$ respectively, (the SEC is larger due to stronger trade winds in the south). In addition, an eastward North Equatorial Counter–Current (NECC) exists in the region of the doldrums, $(4 - 10^\circ N)$. It is highly variable $35 - 60$ cm$^{-1}$ and can extend to 1500 m in depth. To understand these currents, consider the Ekman transport,
The SE trades create a southward Ekman transport in the southern hemisphere but a northward transport between 0°–4°N since $f$ changes sign. In addition from 10°S to the Equator, the sea level is lowered and the thermocline is raised. North of the equator sea level rises because $\tau_x \approx 0$ in the doldrums. Further north the winds again blow to the west and again sea level is raised by the Ekman transport. Geostrophy leads to the observed NECC and the intensification of the SEC and NEC that are associated with the wind driven gyre.
An additional feature is the Equatorial Undercurrent, some 300 km wide and 200m deep extending across the Pacific (14,000 km) with speeds of up to 1.7 ms$^{-1}$ and transport of 40 Sv.

Figure 4.13: Cross sections of temperature, salinity, oxygen, phosphate and velocity at 140°W in the Pacific. In the mixed layer above the thermocline, the water is high in oxygen and low in phosphate. The reverse is true below. Tongues of high salinity extend equatorward. The slope of the thermocline is consistent with a north and south equatorial current separated by a counter current between 5—10°N (confined to the mixed layer). The Equatorial Undercurrent and its effect on the thermocline can be seen at the equator. From Knauss (2000).

The current arises from the westward wind and blocking due to the islands in the Western Pacific. With $x$ in the directed eastward the barotropic momentum equation is given by

$$\frac{du}{dt} - fv = -g \frac{dn}{dx} + \frac{\tau_x}{\rho h} - \frac{ru}{h}.$$
Right at the equator $f = 0$ and if the flow does not accelerate $du/dt = 0$ so that

$$\frac{d\eta}{dx} = \frac{\tau_x}{\rho gh} - \frac{ru}{gh}$$

(4.18)

and water is piled up in the west by the wind. A flow $u$ to the east exists which acts to drain this reservoir of water: the flow is the Equatorial Undercurrent.

The model (Equation 4.18) makes no allowance for how things vary with depth. Within the surface Ekman layer, the current is in the direction of the wind

$$ru \simeq \frac{\tau_x}{\rho} < 0$$

Below this layer, the direct effects of the wind vanish and the Undercurrent is driven by the sea level gradient:

$$\frac{ru}{gh} \simeq -\frac{d\eta}{dx} > 0$$

At depths of more than 300 m, the thermal wind effect due to the depressed isotherms in the western Pacific, cancels out the pressure gradient due to the sea level. The Undercurrent vanishes.

Figure 4.14: Velocity (left) and temperature (right) transect through the equatorial undercurrent. A doming in the isotherms is associated with the eastward flowing undercurrent.

Away from the equator $f$ is non-zero so that a north-south current $v$ may exist. However, north of the equator $f > 0$ so that the geostrophic current $v_g = (g/f) d\eta/dx$ is negative since $d\eta/dx$ is also. Thus, the geostrophic current would tend to push the EUC back along the equator. A similar argument applies to the velocity $v_g$ south of the equator.
Figure 4.15: a) west-east section along the equator of thermosteric anomaly (b,c) Schematic of south-north sections across currents in the western and eastern Pacific (d) south-north section for salinity in west (e) Schematic of south-north section for temperature in the east. Pond and Pickard (1983)
5 Meridional Circulation in the Atmosphere

Since most of the Sun’s energy is absorbed near the equator, the atmosphere in the equatorial belt is warmer and moister (due to higher evaporation rates) than the atmosphere over the polar ice caps. Such horizontal gradients in temperature around the globe induce horizontal pressure gradients and motions. The resulting atmospheric wind patterns (and the resultant ocean currents) transport heat from the warm tropics to the cool high latitudes, thereby playing a major role in climate.

![Diagram of atmospheric circulation](image)

Figure 5.1: The atmosphere is warmer in the equatorial belt than over the polar caps. At high altitudes, the horizontal temperature gradients induce, (by hydrostatic balance), a horizontal pressure gradient force \( P \) that drives air poleward. Conservation of angular momentum induces the air to accelerate eastwards until Coriolis forces acting on them \( C \) are sufficient to balance the pressure gradient force \( P \). *Marshall and Plumb (2007)*

Through the hydrostatic balance, the warmer air at the tropics leads to an ‘expansion’ of tropical air columns relative to colder polar air columns. This change in the vertical height (thickness) of the air columns creates pressure gradients that induce fluid accelerations. This leads to the westerly jetstreams seen in both hemispheres around 1 km above the surface. Because the earth is spinning, this pattern repeats itself meridionally.

5.1 Radiative Forcing

The incoming solar radiation varies with latitude as a consequence of the spherical geometry of the earth and the tilt of the spin axis. The actual radiative budget of the earth depends on the incoming solar radiation as well as the outgoing reflected radiation. The net radiative
budget is shown in Fig 5.2. Averaged over the year there is a net surplus of incoming radiation in the tropics and a net deficit at high latitudes. Since local energy balance must be satisfied, this implies that there must be a transport of energy from low to high latitudes to maintain equilibrium.

Figure 5.2: Annual mean absorbed solar radiation, emitted longwave radiation and net radiation (the sum of the two). *Marshall and Plumb* (2007)

As it is warmer in the tropics than at the poles, this leads to expansion of tropical air columns (through the hydrostatic balance). Hence meridional pressure gradients are established (as described in the previous section). It is these pressure gradients that drive the wind. It
is customary in meteorology to use pressure rather than height as the primary vertical coordinate. Since in hydrostatic balance, pressure is directly related to the overlying mass burden, pressure is actually a mass co-ordinate. In observations it is simpler to measure pressure in situ rather than height. The height of a given pressure surface is dependent on the average temperature below that surface and the surface pressure:

$$z(p) = R \int_{p}^{p_0} \frac{T \, dp}{g \, p}$$

(5.1)

As tropical air columns are warmer than polar columns at all levels, then the pressure tilt must increase with height as in Figure 5.4.

Figure 5.4: Warm columns of air expand, cold columns of air contract, leading to a tilt of pressure surfaces, which typically increases with height in the troposphere. Corresponding winds are out of the page. Marshall and Plumb (2007)

It is also useful to define the thickness of an atmospheric layer, between two pressure surfaces such as $p_1$ and $p_2$, e.g. in Figure 5.4. We have

$$z_2 - z_1 = R \int_{p_2}^{p_1} \frac{T \, dp}{g \, p}$$

(5.2)

which depends on $T$ averaged over the layer. Atmospheric layers are ‘thick’ in tropical regions because they are warm and ‘thin’ in polar regions because they are cold, leading to the large-scale slope of pressure surfaces.

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Figure 5.5: MSLP (solid) and 1000 to 500 hPa thickness (dashed) chart for Australia. The 5400 thickness line (rule of thumb: snow > 1800 m) has penetrated well north over the southeast mainland in cold southwest winds behind a cold front over the Tasman Sea. This brought widespread snow to the NSW and Victorian Alps.

5.2 The Hadley Cell

As tropical air columns are warmer than polar columns at all levels there is thus a pressure gradient aloft directed from high pressure to low pressure, i.e from warm latitudes to cold latitudes. Hadley (1735) suggested that there was one giant meridional cell with rising motion in the tropics and descending motion at the poles. This explanation is a little simplified, and the circulation is more complex.

Since most of the Sun’s energy is absorbed near the equator, we expect greater heating of the atmosphere. This heating results in a Hadley cell whereby the warmed air rises, up to 12 km near the equator, is transported towards the poles and then sinks (at latitudes $\sim 20^\circ - 30^\circ$). This Hadley cell helps to transport the heat towards the colder poles resulting in a more uniform temperature distribution than would otherwise be the case. In addition, the cell drives the westward (easterly) trade winds near the ocean surface.
A simple model for the Hadley cell can be obtained from our equations of motion and from an additional equation for density. This equation may be written as

$$\frac{\partial p}{\partial t} = -w \frac{\partial \rho}{\partial z} - \rho_0 Q - \alpha (\rho - \rho_0)$$  \hspace{1cm} (5.3)

which states that density may change due to 3 factors. The first term (i) involves changes...
in density due to the vertical advection (or transport of density). If \( \rho \) does not change with height \((-z)\), then \( \frac{\partial \rho}{\partial t} = -w \frac{\partial \rho}{\partial z} = 0 \). No change in density. However, suppose that \( \rho \) is larger near the Earth’s surface \((z \simeq 0)\). In this case \( \frac{\partial \rho}{\partial z} > 0 \) and if \( w < 0 \) (upwards motion) then

\[
\frac{\partial \rho}{\partial t} = -w \frac{\partial \rho}{\partial z} > 0 \tag{5.4}
\]

That is the vertical transport of relatively dense air will increase the density above and \( \frac{\partial \rho}{\partial t} > 0 \).

The second term \((ii)\) involves the effect of heating by the long and short waves, and we will assume it can be modelled by

\[
Q = -Q_0 \cos(\ell y) \sin(mz) \tag{5.5}
\]

where \( \ell = \frac{\pi}{L} \) and \( m = \frac{\pi}{D} \). For \( Q_0 > 0 \), this term will heat and cool the air (make less and more dense). The third term \((iii)\) is called Newtonian cooling and acts to make the density \( \rho \) return to some basic state \( p_0(z) \) in the absence of any other effects.

\[
\frac{\partial \rho}{\partial t} = -\alpha (\rho - \rho_0) \tag{5.6}
\]

Using the following decomposition for density \( \rho = \rho_0(z) + \rho'(x, y, z, t) \), the above may be written as

\[
\frac{\partial \rho'}{\partial t} = -\alpha \rho' \]

so if \( \rho' \equiv \hat{\rho} \) at time zero

\[
\rho' = \hat{\rho} \exp(-\alpha t)
\]

and in analogy with friction, \( \rho' \) collapses to zero with an \( e \)-folding time scale equal to \( \frac{1}{\alpha} \).

Now collecting the above, Equation 5.3 may be re-written as

\[
\frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} = -Bw - Q - \frac{\alpha \rho'}{\rho_0} \tag{5.7}
\]

where \( B = \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \) is assumed constant. Our \( x \)-momentum equation is

\[
\frac{\partial u}{\partial t} = fu - Ru \tag{5.8}
\]

where \( R^{-1} \) is the \( e \)-folding time scale for friction. No pressure forces are allowed in the \( x \) (west) direction. Our \( y \)-momentum equation is

\[
\frac{\partial v}{\partial t} = -fu - \frac{1}{\rho_0} \frac{\partial \rho'}{\partial y} - Rv \tag{5.9}
\]
and a pressure force in the southward direction is assumed to arise from the heating $Q(y, z)$. It can be shown that if the system is steady (time independent) such that $\frac{\partial \rho'}{\partial t} = \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0$ and Equations 5.7-5.9 can be simplified, then the solution is:

$$-B \frac{\partial^2 w}{\partial y^2} + \frac{\alpha}{Rg} \left[ f^2 + R^2 \right] \frac{\partial^2 w}{\partial z^2} = \frac{\partial^2 Q}{\partial y^2}. \quad (5.10)$$

At $z = -D$ and $z = 0$, we assume that $w = 0$ and a solution to Equation 5.10 is

$$w = w_0 \cos(\ell y) \sin(mz). \quad (5.11)$$

Substituting this into Equation 5.10 and using $Q = -Q_0 \cos(\ell y) \sin(mz)$ shows that Equation 5.10 is a solution if

$$w_0 = \frac{\ell^2 Q_0}{\ell^2 |B| + \alpha m^2 (f^2 + R^2)} > 0. \quad (5.12)$$

At $y = 0$, $w = w_0 \sin(mz)$ and since $z < 0$, $\sin(mz) < 0$ and $w < 0$. That is the heated air rises. At $y = L$ the reverse holds. The $v$ velocity may be obtained from $\frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$ and is

$$v = v_0 \sin(\ell y) \cos(mz)$$

with

$$v_0 = -\frac{w_0 m}{\ell} < 0.$$

Thus at the ground ($z = 0$), $v = v_0 \sin(\ell y)$ and $v$ is negative. Air is drawn towards the equator. At $z = -D$, the reverse holds.

The trade winds (at $z = 0$) are also set up by the Hadley cell with $u = \frac{vf}{R} > 0$ for $f < 0$. That is the winds blow to the west. A Hadley cell of course exists on both sides of the equation with a circulation as shown in Figure 5.7.
The key features of the circulation may be summarised in the following:

1. The surface winds (away from the equator) blow to the west and are called the trade winds. The warm winds pick up a great deal of moisture over the oceans.

2. Air rises near the surface intersection of the two Hadley cells. This air originates in the region of the trade winds and the water vapour is returned as rain, thunderstorms - the monsoon.

3. The surface intersection of the two Hadley cells is called the intertropical convergence zone (ITCZ) and the trade winds vanish – the doldrums.

4. At a height of about 200mb (10 km) the winds are eastward and are used by jets to enhance ground speed.

5. The surface winds are very important to the ocean circulation and El Niño.

The Hadley cells are not symmetric due to seasonal effects and the presence of land as shown in Figure 5.8. During the Northern winter, the atmospheric temperatures in the northern hemisphere vary more with latitude and the cell circulation is intensified. Over the southern hemisphere summer heating results in a semi-permanent low pressure area over northern Australia and the ITCZ moves into the southern hemisphere. This shift in the wind directions owing to a northward or southward shift in the ITCZ results in the monsoons. Monsoons are wind systems that exhibit a pronounced seasonal reversal in direction. The best known monsoon is found in India and southeast Asia.
Figure 5.8: a) Southern shift of ITCZ in January. b) Northern shift of ITCZ in July. *Lutgens and Tarbuck* (2001),
5.3 Ferrel Cells

5.4 Monsoons

5.5 Trade Winds

5.6 Baroclinic Instability: Highs and Lows

The weather at midlatitudes (40-50 degrees) is dominated by the presence of high and low pressure systems as exemplified by the figure below. What causes these pressure systems and what role do they play in the global redistribution of heat?

*Figure?*

To answer this question (in part) consider the idealised northern hemisphere model sketched below. The atmosphere is cooled by the interior cylinder and heated at the outer cylinder. The interior cooling causes the air to become dense, descend and be deflected to the right as shown. Along the outer cylinder the reverse holds and the air is deflected to the west. In addition, the density field changes as indicated with colder air to the north.

After some time, a thermal wind balance is set up

\[
\frac{\partial v}{\partial z} = \left( \frac{g}{\rho_0 f} \right) \frac{\partial \sigma}{\partial x}
\]

and near the top of the atmosphere, the thermal wind blows in the same direction as the rotating Earth. That is, from west to east. This wind is called the jet stream and is strongest in the northern hemisphere due to the presence of the land masses which accentuate the temperature difference between north and south. In the southern hemisphere, the presence of the Atlantic, Pacific and Southern Oceans moderate temperature difference.
The high and low pressure systems result from a process known as baroclinic instability. The energy for this instability (and the pressure systems) comes from the tilted density field (shown above) that results from the cooling and heating. The instability, illustrated below, “feeds” on this energy and the resultant high and low systems act to transport heat away from the equator and towards the poles.

5.7 The Sea Breeze

The afternoon sea breeze is caused by the differential heating by land and sea that can be quite strong during summer. The warmer land acts to heat the air above the land leading to lighter air and low in pressure. The denser, cooler air over the ocean slides under and towards the land resulting in the breeze. The vertical motion associated with the breeze is also effected by the change in the density of air.

Over the land, the hydrostatic balance does not hold. Rather, since the air is less dense, the upwards pressure force $\frac{1}{\rho} \frac{dp}{dz}$ exceeds that due to gravity $g$ and

$$\frac{dw}{dt} = \frac{1}{\rho} \frac{dp}{dz} + g < 0,$$

so that $w$ is negative (rising air). Over the sea the reverse holds.
References


