Strictly singular and compact operators

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Given Banach spaces E and F, a linear operator $A : E \to F$ is called strictly singular if the restriction of A to any infinite–dimensiinal subspace of E is not an isomorphism. This consept was introduced by T. Kato. Strictly singular operators form an operator ideal which contains the ideal of compact operators. We will denote by $\mathrm{SS}(L_p)$ ($K(L_p)$) the set of strictly singular (compact) operators in $L_p = L_p[0,1]$. It is well known that $\mathrm{SS}(L_p) = K(L_p)$ iff $p = 2$.

Theorem 1. *Let $1 < q < r < \infty$ and A be a linear operator bounded in L_q and L_r. Then one of the following alternatives holds:*

1. *$A \in K(L_p)$ for any $p \in (q, r)$, or*
2. *$A \overline{\in} \mathrm{SS}(L_p)$ for any $p \in (q, r)$.*

Let $1 \leqslant p \leqslant \infty$. Given a bounded linear operator A in L_p, denote $O(A) = \{q : 1 \leqslant q \leqslant \infty, A \text{ is bounded in } L_q\}$. By M. Riesz interpolation theorem $O(A)$ is a convex subset R^1. This subset may be open, closed or semiclosed. Denote $V_p = \mathrm{SS}(L_p) \setminus K(L_p)$.

Theorem 2. *Let $1 < p < \infty$, $p \neq 2$ and $A \in V_p$. Then p is endpoint of $O(A)$. Moreover, if $p > 2$, then p is the right endpoint of $O(A)$, if $p < 2$, then p is the left endpoint of $O(A)$.*

All known examples of operators in V_p always depend on p. The following theorem explains this phenomenon.

Theorem 3. *Let $1 < q < p < \infty$. The set $V_p \cap V_q$ is not empty iff $q < 2 < p$.*

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