Characters, bi-modules and representations in Lie group harmonic analysis

N.J. Wildberger
School of Mathematics
University of New South Wales
Sydney, 2052
AUSTRALIA

August 28, 2001

Abstract

This paper is a personal look at some issues in the representation theory of Lie groups having to do with the role of commutative hypergroups, bi-modules, and the construction of representations. We begin by considering Frobenius' original approach to the character theory of a finite group and extending it to the Lie group setting, and then introduce bi-modules as objects intermediate between characters and representations in the theory. A simplified way of understanding the formalism of geometric quantization, at least for compact Lie groups, is presented, which leads to a canonical bi-module of functions on an integral coadjoint orbit. Some meta-mathematical issues relating to the construction of representations are considered.

1 Introduction

This paper describes an approach to non-commutative harmonic analysis on a Lie group G which is based on an old idea of Frobenius. We discuss the possible role of commutative hypergroups (in the same vein as [19], [24]), introduce G bi-modules and consider a computational approach to the construction of irreducible representations. Most of the ideas are of an elementary nature. Along the way certain amusing but perhaps unsettling philosophical points are raised.

This is a personal approach– the opinions expressed are those of the author, and occasionally diverge from the mainstream view. While there is then the increased likelihood of saying something foolish, there is the advantage of encouraging discussion. Good natured debate is a sign of vigour in a theory, and it is in this spirit that I offer this paper.

Harmonic analysis on a non-commutative finite group G was initiated one hundred years ago by G. Frobenius in a paper entitled 'Über Gruppencharaktere' [8]. In this fundamental work Frobenius introduced the idea of a character and computed some character tables.

Nowadays we consider characters useful objects associated to representations, which were not mentioned in Frobenius' 1896 paper. They were introduced by him shortly thereafter when the modern definition appears: one starts with a representation $\pi : G \to Gl(n)$ and defines the associated character to be the function $\chi(g) = \text{ tr } \pi(g)$. Our attitude is captured by the following quote of G. Mackey [13].

> Frobenius' original definition of character was a complicated one, which emerged from his analysis of Dedekind's problem. A year later, however, he showed that his definition is equivalent to another that is much simpler and more natural.

In other words, the fundamental question today is not

Question 1) What are all the characters of a group G?

as it was for Frobenius in 1896 but rather

Question 3) What are all the equivalence classes of irreducible representations of G?

In my opinion, Frobenius' original point of view towards characters deserves to be reconsidered as possibly more fundamental than the subsequent modern approach. From this standpoint, Question 1 becomes the natural starting point for harmonic analysis on a group G and Question 3 logically follows it. Between these is another reasonable question which seems to have attracted little attention.

Question 2) What are all the equivalence classes of irreducible G bi-modules?

Frobenius' 1896 approach can be extended to other families of groups by utilizing the concept of a locally-compact commutative hypergroup and possible generalizations. The central object of interest is the *class hypergroup* $\mathcal{K}(G)$ of conjugacy classes of a group G.

For a compact Lie group G, $\mathcal{K}(G)$ is a locally compact, compact hypergroup in the sense of Dunkl [7], Jewett [11] and Spector [15]. There is a basic relation between this hypergroup and the hypergroup of adjoint orbits $\mathcal{K}(\mathfrak{g})$ in the Lie algebra $\mathfrak{g}$ of G given by the *wrapping map* (see [6]). From this vantage point many important aspects of harmonic analysis are seen in a new and simplified light. In particular, the classification of irreducible representations by highest weights or integral coadjoint orbits, the Kirillov character formula [12], a formula of Harish-Chandra on G-invariant differential operators, the Duflo isomorphism, the Poisson-Plancherel formula of M. Vergne [18] and the identity of Thompson [17], amongst other things, can be explained both naturally and easily.

For non-compact Lie groups, the definition of $\mathcal{K}(G)$ is more subtle and problematic. At this point, little is known about the right way of proceeding in general although there is evidence that a suitable form of the wrapping map should apply in some situations. What seems clear, however, is that the present theory of locally compact hypergroups is too narrow to accomodate objects which arise as quotients of non-compact group actions. We need to understand hypergroups based on spaces which are not necessarily Hausdorff or locally compact. Its seems a project of some importance to establish such a theory and apply it to the harmonic analysis of non-compact Lie groups. Such a development is also likely to be of interest to mathematical physicists.

In section 2, we define hypergroups, describe Frobenius' original approach to characters, and indicate how to extend this to a compact Lie group. In section 3, we introduce the notion of G bi-modules and their potential role as intermediate objects between characters and representations. In section 4 the natural occurence of such G bi-modules for a compact Lie group G as spaces of functions on coadjoint orbits and the connection with moment maps and geometric quantization is described. The last section raises the possibility that the problem of 'constructing' representations of groups has been obscured by the lack of precision in usage of words like 'construct'. I propose a reasonably concrete meaning for this term and suggest that the task of constructing the irreducible representations of a compact Lie group is by no means completed.

Some problems of interest are scattered through the paper.

2 Hypergroups and Frobenius' approach to characters

The concept which clarifies Frobenius' paper is that of a *finite commmutative hypergroup.* This notion is almost implicit in Frobenius' 1896 work. In retrospect, it seems curious that in a century of mathematics oriented towards abstract algebra, this important theory has been developed only recently. Since we will be interested in applications to Lie groups we give a more general definition without a precise discussion of the topologies involved- see [11] or [4] for more detail.

DEFINITION 1 *A* locally compact commutative hypergroup *is a locally compact space $\mathcal{K}$ for which the Borel measures $M(\mathcal{K})$ form a $*$-algebra satisfying essentially the following axioms.*

1) (Closure) The product of Dirac delta functions $\delta_x \times \delta_y$, for $x, y \in \mathcal{K}$, is always a compactly supported probability measure which varies continously with x and y.

2)(Associativity) The algebra $M(\mathcal{K})$ is associative.

3)(Existence of an Identity) There exists an element $e \in \mathcal{K}$ such that δ_e is the identity.

4)(Existence of Inverses) For every $x \in \mathcal{K}$ there exists a unique element $x^* \in \mathcal{K}$ such that e is contained in the support of the measure $\delta_x \times \delta_{x^*}$. Furthermore $(\delta_x)^* = \delta_{x^*}$.

5)(Commutativity) The algebra $M(\mathcal{K})$ is commutative.

In the case of a finite set $\mathcal{K} = \{c_0, c_1, \cdots, c_n\}$, this definition coincides, once we identify measures and functions, with the definition given in the paper [16] in this same volume. A finite commutative hypergroup is as close to being a commutative group as possible given that we only require the product of two elements to be a probability distribution of elements; in particular a commutative group is also a commutative hypergroup. The notions of character, duality and Fourier transform for commutative groups extend to finite commutatve hypergroups once we consider also somewhat more general objects called *signed hypergroups* which involve negative probabilities.

Frobenius' original approach may now be restated into modern language as follows. For any non-commutative finite group G, there is an associated finite commutative hypergroup $\mathcal{K}(G)$, called the *class hypergroup* of G, obtained from convolving G-invariant probability measures supported on conjugacy classes. The characters of the hypergroup $\mathcal{K}(G)$ form a signed hypergroup $\mathcal{K}(G)^\wedge$ which happens to be a hypergroup. The elements of this dual hypergroup are functions on $\mathcal{K}(G)$ and so can be naturally interpreted as central functions on G. They are precisely the irreducible characters in the usual sense, except that they have been normalized to have value 1 at the identity.

A number of facts about characters of finite groups such as the orthogonality relations and integrality of character values are special cases of more general facts which hold for finite commutative hypergroups (for some deeper examples, see [2], where the terminology 'table algebras' are used). This is useful since there are many examples of finite commutative hypergroups which arise outside of group theory, for example in the theory of distance regular graphs, association schemes, conformal field theory, cyclotomy, inclusions of Von Neumann algebras etc see [20].

Summarizing, we may say that the problem of determining the characters of a non-commutative group G is essentially a problem of commutative harmonic analysis; analysis on the associated class hypergroup $\mathcal{K}(G)$.

Let us now try to generalize this approach to the case of G a compact Lie group. Let $\mathcal{K} = \mathcal{K}(G)$ be the set of conjugacy classes of G, which we may view as the quotient of G with respect to the conjugation action on itself. Measures on $\mathcal{K}(G)$ form an algebra under convolution induced by the convolution algebra of central measures on G. In this correspondence, delta functions on $\mathcal{K}(G)$ are associated to invariant probability measures on conjugacy classes.

A character of $\mathcal{K}(G)$ is a bounded continuous function $\chi : \mathcal{K}(G) \to \mathbb{C}$ such that

1) $\chi(x)\chi(y) = \int_{\mathcal{K}(G)} \chi(z) d(\delta_x \times \delta_y)(z)$ for all $x, y \in \mathcal{K}$

2) $\chi(x^*) = \overline{\chi(x)}$ for all $x \in \mathcal{K}$.

The set of characters of a hypergroup $\mathcal{K}$ is denoted by $\mathcal{K}^\wedge$. A character of $\mathcal{K}(G)$ lifts via the quotient map $p : G \to \mathcal{K}(G)$ to a function on G invariant on conjugacy classes.

Definition 2 *Any function on G obtained this way from a character of $\mathcal{K}(G)$ is called an* irreducible normalized character *of G.*

It is a consequence of a well known theorem of Weyl that this notion is exactly the same as the usual one; that is, an irreducible normalized character is exactly a function of the form $\chi(g) = \frac{1}{n} \operatorname{tr} \pi(g)$ for some irreducible representation $\pi : G \to Gl(n)$.

Our stategy towards understanding harmonic analysis on a compact Lie group G is now the following. The first step is to understand $\mathcal{K}(G)$ and its hypergroup structure– this is the basic object. The next step is to determine the characters of $\mathcal{K}(G)$ and the hypergroup structure of $\mathcal{K}(G)^\wedge$. The next step is to construct as explicitly as possible the irreducible bi-modules of G and the irreducible representations of G.

The first two steps can be accomplished by turning our attention to the Lie algebra $\mathfrak{g}$ and to certain hypergroup structures on it and its dual $\mathfrak{g}^*$. These hypergroup structures are special cases of a general phenomenon: for any linear action of G on a vector space V, the space of orbits of G on V carries a hypergroup structure obtained by convolving G-invariant probability measures in $\mathfrak{g}$.

The resulting orbit hypergroup $\mathcal{K}(V; G)$ has dual the orbit hypergroup $\mathcal{K}(V^*; G)$. Here G acts on the dual vector space as follows:

$$g \cdot f(v) = f(g^{-1} \cdot v) \text{ for all } g \in G, f \in V^*, v \in V.$$

Furthermore the pairing between the G orbit $\mathcal{U}$ of V and the G orbit $\mathcal{O}$ of V^* is given by:

$$\langle \mathcal{U}, \mathcal{O} \rangle = \int_{\mathcal{U}} \int_{\mathcal{O}} e^{if(v)} d\mu_{\mathcal{O}}(f) d\mu_{\mathcal{U}}(v) \tag{2.1}$$

where $d\mu_{\mathcal{O}}$ and $d\mu_{\mathcal{U}}$ are the G-invariant probability measures on $\mathcal{O}$ and $\mathcal{U}$.

Since G acts on $\mathfrak{g}$ and on $\mathfrak{g}^*$ by the adjoint and coadjoint actions respectively, we have hypergroups $\mathcal{K}(\mathfrak{g}\ ; G)$ and $\mathcal{K}(\mathfrak{g}^*; G)$ which are in duality.

To understand the connection between the class hypergroup $\mathcal{K}(G)$ and the *adjoint hypergroup* $\mathcal{K}(\mathfrak{g}\ ; G)$ we now introduce the wrapping map Φ from distributions of compact support on $\mathfrak{g}$ to distributions on G. The consideration of this map is motivated by the work of Harish-Chandra, Helgason, Kashiwara- Vergne, and Duflo. Let j denote a suitably chosen square root of the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$. This function is analytic on $\mathfrak{g}$, has value 1 at 0 and is G-invariant (with respect to the adjoint action). For a function φ on G let $\tilde{\varphi} = \varphi \circ \exp$ be its lift to $\mathfrak{g}$. For a distribution μ of compact support on $\mathfrak{g}$, define the distribution $\Phi(\mu)$ on G by

$$\langle \Phi(\mu), \varphi \rangle = \langle \mu, j\tilde{\varphi} \rangle$$

for any $\varphi \in C^\infty(G)$.

The following result, which we call the *wrapping theorem*, (see [6]) is a generalization of results of Harish-Chandra, Duflo and I. Frenkel.

THEOREM 3 *Let μ and ν be two G-invariant distributions of compact support on $\mathfrak{g}$. Then*

$$\Phi(\mu) * \Phi(\nu) = \Phi(\mu * \nu)$$

where the convolution on the left is group convolution on G and the convolution on the right is Euclidean convolution in $\mathfrak{g}$.

The wrapping theorem allows us to relate the structures of the hypergroups $\mathcal{K}(G)$ and $\mathcal{K}(\mathfrak{g}\ ; G)$. This explains why Kirillov's orbit theory of representations works. Coadjoint orbits and representations are intimately related since the former determine characters of the adjoint hypergroup and the latter determine characters of the class hypergroup. The character formula of Kirillov follows from the pairing of adjoint and coadjoint orbits. For a more extensive discussion of the applications of this point of view to compact Lie groups, see [6] , [24].

How does any of this generalize to non-compact Lie groups? This is not at all so clear, since the notion of a G-invariant probability measure on a conjugacy class in general is not defined, and even if it were, how would one convolve two such things? There are some indications that an appropriate theory of means could be used for some groups at least (see [19]). The fact that much of Kirillov theory extends to non-compact groups suggests that such an approach ought to exist. We know what the answer is– the theory of characters of non-compact Lie groups as developed in the nilpotent case by Kirillov and in the semisimple case by Harish-Chandra and others; the question is – how to recover these results from a hypergroup approach? In other words, how can one obtain the results of representation theory on characters for non-compact Lie groups without mentioning representations? The following is thus a key project.

Problem 1) Develop a more general theory of hypergroups into which the class hypergroups of non-compact Lie groups may fit.

3 G bi-modules

In this section we introduce a family of objects which perhaps can play a role in harmonic analysis somewhere between characters and representations. The notion of a bi-module is familiar enough in other areas of mathematics, for example the theory of Von Neumann algebras where it has been introduced by Connes.

Let G be a Lie group; that is a smooth manifold with a compatible smooth group structure.

DEFINITION 4 *A* left-action *of G on a finite dimensional vector space V is an assignment to each $g \in G$ and $v \in V$ an element $g \cdot v \in V$ such that*

1) $g \cdot v$ varies smoothly with g and v

2) $g \cdot v$ varies linearly with v

3) $g \cdot (h \cdot v) = (gh) \cdot v$ for all $g, h \in G, v \in V$.

We will also say that G *left-acts* on V, or that V is a G *left-module*.

DEFINITION 5 *A* right-action *of G on a finite-dimensional vector space V is an assignment to each $g \in G$ and to each $v \in V$ an element $v \cdot g \in V$ such that*

1) $v \cdot g$ varies smoothly with g and v

2) $v \cdot g$ varies linearly with v

3) $(v \cdot g) \cdot h = v \cdot gh$ for all $g, h \in G, v \in V$.

In this case we say that V is a G *right-module*. Notice that if V is say a G left-module then V^* can be made into a G right-module by defining

$$(f \cdot g)\ (v) = f(g \cdot v) \text{ for all } f \in V^*, g \in G, v \in V.$$

Definition 6 *A vector space W is a G* bi-module *if it is both a G left-module and a G right-module and if these actions are* compatible*, that is if*

$$g \cdot (w \cdot h) = (g \cdot w) \cdot h \text{ for all } g, h \in G, w \in W.$$

Definition 7 *A G bi-module W is* irreducible *if there is no proper subspace $W_0 \subset W$ which is stable under both actions.*

Definition 8 *A G bi-module W is* symmetric *if there exists an involution (a linear map whose square is the identity) $^* : W \to W$ such that*

$$(g \cdot (w \cdot h))^* = (h^{-1} \cdot w^*) \cdot g^{-1}$$

If V is a G left-module then V^* is a G right-module and the tensor product $V \otimes V^*$ is a G bi-module in the obvious way. If in addition we are given a map $^* : V \to V^*$ such that

$$(g \cdot v)^* = v^* \cdot g^{-1}$$

then $V \otimes V^*$ becomes a symmetric G bi-module by defining

$$(v \otimes u^*)^* = u \otimes v^*.$$

This is the situation when a G left-action on V preserves a bilinear symmetric form $(\ , \ )$ on V.

Problem 2) For a given group G, construct its irreducible G bi-modules (up to the natural notion of equivalence), its irreducible symmetric bi-modules etc.

Remark 1) The requirement of smoothness in the definition of a left-action is not standard. In the literature smoothness is usually replaced by continuity. The reasons are perhaps that
a) for compact Lie groups the two notions happen to coincide, and

b) requiring only continuity allows an immediate extension of the definition to infinite dimensional vector spaces.

Quite frankly, this seems to me an unfortunate sleight of hand. Functorial and aesthetic considerations urge us to respect the smooth structure of a Lie group. This means taking care to investigate the meaning of smooth structures on infinite dimensional vector spaces before we extend our definitions to that realm. Since representation theory is of such interest and usefulness, it is appropriate that the basic definitions be natural and functorial.

Remark 2) If one takes a Lie group to mean analytic manifold with analytic group operations, then the above definitions should be modified appropriately. Perhaps harmonic analysis on a smooth Lie group is quite a different subject from harmonic analysis on an analytic Lie group. Note that on the latter, distribution theory is sometimes not naturally available– on a non-compact analytic Lie group for example, there are no naturally occurring functions of compact support!

Remark 3) The terminology of left-action and right-action motivates us to constantly acknowledge the presence of any assymetry which we introduce into the situation. This is a another 'philosophical' point which I believe is of some general usefulness. It is most pleasant when our basic concepts are free of arbitrary bias, even the seemingly innocent one of left vs right. In this context, does anyone know how to define the Lie algebra of a Lie group in a completely symmetric way?

Nevertheless, we will continue to follow standard usage and interchange the terms left-action and representation.

Why should we consider G bi-modules? The reason is that they occur naturally. The most obvious representation of any finite group is the regular representation on the space of all functions on the group. This is of course naturally a G bi-module, since we may translate functions on the left or on the right. As a G bi-module, this space decomposes into irreducible G bi-modules, one for each of the irreducible representations of the group. Each such constituent W is a two sided ideal of the convolution algebra of all functions and contains the character of the representation as an minimal idempotent. As a left-module, W decomposes into $\dim V$ copies of an irreducible left-module V, but the decomposition is not canonical. This is a recurring theme in harmonic analysis and quantization theory; the canonical 'square' object which has no natural decomposition into 'left' or 'right' objects. Here we are examining the 'square' objects in their own right.

Given a representation of G on a space V, there is an obvious G bi-module structure on $W = \text{End } V \simeq V \otimes V^*$, which is in addition an algebra. Equivalently we may obtain W by considering the space of all matrix coefficients of the representation. The other direction, from W to V, is not at all so easy and is rarely canonical– which is why the bi-module W perhaps qualifies as a simpler object than the representation V.

4 Geometric quantization revisited

As an illustration of the occurrence of G bi-modules in harmonic analysis, let us return to the situation of a compact Lie group G and consider the method of geometric quantization which associates to an integral coadjoint orbit $\mathcal{O}$ an irreducible representation π of G, that is, the Borel- Weil theorem. We briefly review this theory (see [10], [14], [25],) and then sketch a simpler way of understanding it by considering moment maps of representations. The possibility of such a simplification was suggested to the author by a conversation with I. Frenkel. The presence of a canonical G bi-module of functions on $\mathcal{O}$ follows from this approach.

The assumption that $\mathcal{O}$ is integral means that it carries a complex structure (not unique) such that there exists a holomorphic Hermitian line bundle L over $\mathcal{O}$ with connection whose curvature is equal to the canonical symplectic form on $\mathcal{O}$ (all coadjoint orbits are symplectic manifolds). There is a formula involving the covariant derivative of the connection that shows that the Lie algebra $\mathfrak{g}$ acts naturally on the space of all sections of this bundle. Equivalently one may rephrase this in terms of a circle bundle over $\mathcal{O}$ and spaces of functions on this bundle that transform correctly under the S^1 action (see [14] for more details). The irreducible module associated to the orbit is obtained by taking the submodule of holomorphic sections. All irreducible representations of G occur in this way.

The procedure is admittedly somewhat magical, but there is a more conceptual way of understanding it in which the objects and constructions are more natural. The key is to emphasize the inverse procedure of *dequantization* (see [5], [21]).

Dequantization in this context means going backwards from an irreducible unitary representation $\pi : G \to U(V)$ to the coadjoint orbit $\mathcal{O}$ and its associated geometric objects. Here $U(V)$ is the group of unitary operators on some complex inner product space V. Let Ω be the unit sphere of V. We consider the map $\phi : \Omega \to \mathfrak{g}^*$ defined by

$$\phi(v)(X) = \frac{1}{i} \langle d\pi(X)v, v\rangle$$

for $v \in \Omega$ and $X \in \mathfrak{g}$. The map ϕ is the composition of the projection of Ω onto the projective space PV of V and the moment map of the representation ([9], [21], [22], [23]).

The image of ϕ is a G-invariant compact subset of $\mathfrak{g}^*$ whose convex hull has extremal set a single coadjoint orbit $\mathcal{O}$. In fact most of the time the image of ϕ is convex , see [1] and [23] for the exact statement. In any case the preimage of this extremal orbit $\mathcal{O}$ is a single G orbit $\mathcal{M}$ in Ω and $\phi : \mathcal{M} \to \mathcal{O}$ is a circle bundle; the S^1 action is the restriction to $\mathcal{M}$ of the natural multiplicative action of S^1 on V. The orbit $\mathcal{O}$ is the same coadjoint orbit that geometric quantization utilizes in the construction of π; $\mathcal{M}$ is actually the orbit of the highest weight vector of the representation and the circle bundle over $\mathcal{O}$ is the same as that constructed by geometric quantization. The holomorphic and Hermitian structure of the associated line bundle L follow directly from the holomorphic and Hermitian structure of the space V.

There is furthermore a natural way of assigning to any vector $v \in V$ a function f_v on $\mathcal{M}$ by the rule $f_v(w) = \langle v, w\rangle$ for $w \in U$. This function does not push down to $\mathcal{O}$ but nevertheless has exactly the correct transform properties to identify it with a section of the corresponding line bundle L. This realizes V as a space of sections of L on which G acts.

We now claim that there is a space of functions on $\mathcal{O}$ which carries the G bi-module structure of End V. To an operator $T \in \text{End } V$ associate the function $\sigma_T : \mathcal{M} \to \mathbb{C}$ defined by

$$\sigma_T(v) = \langle Tv, v\rangle.$$

Since this function is independent of the phase of v, it *is* the lift of a function on $\mathcal{O}$ which we denote by a_T. Let A denote the set of all such functions on $\mathcal{O}$. It is then a fact (see [21]) that the assignment $T :\to a_T$ is $1:1$. Recall that an operator T on a complex vector space is determined by its expectation values $\langle Tv, v\rangle$ as v ranges over the unit sphere. We are stating that this is still true when the sphere is replaced by the G orbit $\mathcal{M}$.

Thus we have a canonical identification of End V with the space of functions A on $\mathcal{O}$.

Problem 3) Identify the space A geometrically.

Problem 4) How does the G bi-module structure of A, which it inherits from the fact that End V is a G bi-module, manifest itself in terms of the geometry of the action of G on $\mathcal{O}$?

A related question, which ties in with the theory of $*$ -products (see [3], [5]) is

Problem 5) How does one describe the algebra structure of A, also inherited from End V as above, in terms of the geometry of $\mathcal{O}$?

Until these questions have been answered, the 'construction' of the G bi-module A is admittedly abstract, but it is of some interest perhaps that we need only functions on the coadjoint orbit, not sections of a line bundle, to exhibit the space. For the case of $G = SU(2)$ and the irreducible representation of dimension n we may be more concrete. The space A turns out to be all spherical harmonics of degree up to and including $n-1$ on the sphere of radius $n-1$; a space of dimension n^2.

Problem 6) Develop an analogous theory for the discrete series of a non-compact semisimple Lie group.

5 Construction of Representations

The construction of irreducible unitary representations of a Lie group G is an important problem in non-commutative harmonic analysis. For compact and nilpotent Lie groups, this problem is treated as solved in the literature. It is considered unsolved, for example, for noncompact semisimple groups.

But let us stop for a moment and ask– What does it actually *mean* to 'construct a representation'? This is a meta-mathematical question, at least to the extent that one rarely finds a proper definition of the term 'construct' in the literature.

To clarify the discussion, consider once again the case of a compact simple Lie group G. Weyl determined the irreducible characters of such a group in the 1930's. He was certainly aware of the following 'method' of constructing a representation from a character.

1) Determine the space W of functions on G spanned by all left and right translates of the given character– this is a G bi-module.
2) Decompose W when viewed as a G left-module into irreducibles– we get dim V copies of a single irreducible V.
3) Take any one of the constituents– this is the required left-module.

Is this a valid construction? Well, no, since the credit for the construction of the representations goes to the Borel-Weil theorem, which only appeared some decades later. But this is a sociological answer, not a mathematical one. Mathematically, one would argue that the instructions 1), 2) and 3) are not specific enough. To what extent is the space W constructed by declaring it to be the span of all left and right translates of a character? How exactly does one decompose this bi-module into irreducibles? Can one exhibit in an explicit fashion some non-zero vector in the final representation space? A basis?

These seem quite reasonable objections, but cannot similar queries be raised about the concreteness of geometric quantization? How does one explicitly construct a line bundle over some integral coadjoint orbit? And what does this mean? How does one determine the holomorphic sections of such a bundle? Can one exhibit in an explicit fashion some non-zero vector of the final representation space? A basis?

There is some scope for disagreement and controversy here. Let us therefore try to define more precisely what we might mean by the meta-mathematical term 'to construct a representation'.

I tentatively propose the following:

DEFINITION 9 *A (finite-dimensional) representation of a group G is* constructed *if a computer program can be exhibited which will*
1) input group elements (in whatever form the group has been 'given')

2) output matrices which represent those group elements in some arbitrary but fixed basis of the representation space.

Of course it is reasonable to require only that explicit instructions for creating such a program be given, not the program itself, at least if we can agree that the instructions are indeed explicit enough. (One should also give some thought to how one 'represents' the real numbers which might appear as the matrix entries). Similarly we will say that one has constructed all the representations of a group when one has a (larger) program which in addition to the above, also inputs the representation (by some label such as the highest weight, or the integral coadjoint orbit $\mathcal{O}$). One has constructed the representations of all simple compact Lie groups if one has a (yet larger) program that in addition to the above, inputs the group G.

I realize that this proposal will not give universal pleasure. Are there any sensible alternatives? I suspect that some physicists view our abstract constructions of representations as not completely the full story– witness the large number of papers in physics journals concerned with describing explicit bases of representation spaces. It would be unfortunate if young mathematicians miss out on a chance to contribute to this area of physics because of an imprecision of terminology.

If we accept the above tentative definition, some interesting problems present themselves. Primary among them is the following.

Problem 7) Construct the irreducible unitary representations of a compact simple Lie group G.

Here are some others.

Problem 8) Given an irreducible representation of G on V, describe the geometry of the orbits of G on V.

Problem 9) How does one do *trigonometry* on such orbits?

Problem 10) Describe the hypergroup structure of the G orbits on V.

References

[1] D. Arnal and J. Ludwig, *La convexité de l'application moment d'un groupe de Lie*, J. Funct. Anal. 105 (1992), 256-300.

[2] Z. Arad and H. Blau, *On table algebras and applications to products of characters*, J. of Alg. 138 (1991), 186-194.

[3] F. Bayen, C. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation Theory and Quantization I. Deformations of symplectic structures*, Annals of Physics 111 (1978), 61-110.

[4] W. R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, De Gruyter, Berlin, (1995).

[5] M. Cahen, S. Gutt and J. Rawnsley, *Quantization of Kähler manifolds. I. Geometric interpretation of Berezin's quantization*, J. Geom. Phys. 7 (1990), no.1, 45-62.

[6] A. H. Dooley and N. J. Wildberger, *Harmonic analysis and the global explonential map for compact Lie groups*, (Russian) Funktsional. Anal. i Prilozhen. 27 (1993), no. 1, 25-32 tranlsation in Functional Anal. Appl. 27 (1993), no. 1, 21-27.

[7] C. F. Dunkl, *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. 179 (1973), 331-348.

[8] G. Frobenius, *Über Gruppencharaktere*, Gesammelte Abhandlungen, Vol. III, Springer-Verlag, Berlin, 1-37.

[9] V. Guillemin and S. Sternberg, Symplectic techniques in physics, Cambridge University Press, 1984, Cambridge.

[10] N. E. Hurt, Geometric Quantization in Action, Mathematics and its Applications Vol. 8, Reidel, 1983, Dordrecht.

[11] R. I. Jewett, *Spaces with an abstract convolution of measures*, Adv. Math. 18 (1975), 1-101.

[12] A. A. Kirillov, *Elements of the Theory of Representations*, Grundlehren der math. Wissenschaften 220, Springer-Verlag, Berlia, 1976.

[13] G. Mackey, *Harmonic analysis as exploitation of Symmetry*, Bull. Amer. Math. Soc. 3 number 1 (July, 1980), 543–698.

[14] J. Sniatycki, Geometric Quantization and Quantum Mechanics, 1980, Springer-Verlag, New York.

[15] R. Spector, *Mesures invariantes sur les hypergroupes*, Trans. Amer. Math. Soc. 239 (1978), 147-165.

[16] V. S. Sunder and N. J. Wildberger, *On discrete hypergroups and their actions on sets*, preprint (1996).

[17] R. Thompson, *Author vs Referee: A case history for middle level mathematicians*, Amer. Math. Monthly 90 No. 10 (1983), 661-668.

[18] M. Vergne, *A Plancherel formula without group representations*, in Operator Algebras and Group Representations Vol. II, Neptune, (1980), 217-226.

[19] N. J. Wildberger, *Hypergroups and Harmonic Analysis*, Centre Math. Anal. (ANU) 29 (1992), 238-253.

[20] N. J. Wildberger, *Finite commutative hypergroups and applications from group theory to conformal field theory*, Contemp. Math. 183 (1995), 413-434.

[21] N. J. Wildberger, *On the Fourier transform of a compact semisimple Lie group*, J. Austral. Math. Soc. (Series A) 56 (1994), 64-116.

[22] N. J. Wildberger, *Convexity and representations of nilpotent Lie groups*, Invent. Math. 98 (1989), 281-292.

[23] N. J. Wildberger, *The moment map of a Lie group representation*, Trans. Amer. Math. Soc. 330 (1992), 257-268.

[24] N. J. Wildberger, *Hypergroups, Symmetric spaces, and wrapping maps*, in Probability Measures on Groups and related structures, Proc. Oberwolfach 1994, World Scientific, Singapore, 1995.

[25] N. M. J. Woodhouse, *Geometric Quantization*, Clarendon Press, 1992, Oxford.