How well-connected is the surface of the global ocean?

Gary Froyland,1 Robyn M. Stuart,1 and Erik van Sebille2
1) School of Mathematics and Statistics, The University of New South Wales, Sydney NSW 2052, Australia
2) Climate Change Research Centre and ARC Centre of Excellence for Climate System Science, The University of New South Wales, Sydney NSW 2052, Australia

(Dated: 14 April 2014)

The Ekman dynamics of the ocean surface circulation is known to contain attracting regions such as the great oceanic gyres and the associated garbage patches. Less well-known are the extents of the basins of attractions of these regions and how strongly attracting they are. Understanding the shape and extent of the basins of attraction sheds light on the question of the strength of connectivity of different regions of the ocean, which helps in understanding the flow of buoyant material like plastic litter. Using short flow time trajectory data from a global ocean model, we create a Markov chain model of the surface ocean dynamics. The surface ocean is not a conservative dynamical system as water in the ocean follows three-dimensional pathways, with upwelling and downwelling in certain regions. Using our Markov chain model we easily compute net surface upwelling and downwelling, and verify that it matches observed patterns of upwelling and downwelling in the real ocean. We analyze the Markov chain to determine multiple attracting regions and their absorption probabilities. Finally, using an eigenvector approach, we (i) identify the five major ocean garbage patches, (ii) partition the ocean into basins of attraction for each of the garbage patches, and (iii) partition the ocean into regions which are approximately forward-invariant (almost-invariant sets).

Keywords: Ocean currents, attracting set, basin of attraction, almost-invariant set, Markov chain, transfer operator

Ocean dynamics operate and affect climate on timescales of months to millennia. In this paper we investigate phenomena on the ocean’s surface that manifest over very long time periods: we look for regions in which water, biomass and pollutants become trapped “forever” (which we refer to as attracting regions), or for long periods of time before eventually exiting (which we refer to as almost-invariant regions). While attracting regions may be quite small in size or irregular in shape, they can nonetheless exert great influence on the global ocean surface dynamics if their basins of attraction are large.

I. INTRODUCTION

In this paper we study how well water mixes between different regions of the surface ocean. A better understanding of the surface ocean’s mixing properties might help us study the evolution of the so-called great ocean garbage patches1–4, which are regions in which plastics and other floating debris accumulate after being carried there by winds and currents.

Several recent papers have analyzed almost-invariant regions in oceans5–7, but less attention has been devoted to analyzing attracting regions and their basins of attraction. One study in this direction is Kazantsev8,9, who identified low-period orbits for a barotropic ocean model on a square.

In this paper we take a probabilistic approach, which is based on analyzing a spatial discretisation of the flow dynamics. We take a set of short-run trajectories from a global ocean model and use these to construct a transition matrix, thereby representing the dynamics as a Markov chain (see4–6); a related approach, which may be viewed as a time-derivative of this construction is discussed in10,11. The transition matrix enables us to efficiently compute the evolution of densities and to calculate surface upwelling and downwelling. We are also able to make probabilistic statements about the flow; in particular, we can define the probability of eventual absorption into an attracting region from any other region of the surface ocean.

We identify attracting regions of the surface ocean and calculate the probability of being eventually absorbed into one of these regions. We then use the absorption probabilities as a means of partitioning the ocean into different non-overlapping regions, by calculating into which attracting set a given particle is most likely eventually absorbed. These results are then compared with another criterion for partitioning the ocean, based on identifying regions that interact minimally with other regions. The transfer operator methods developed in12 reveal the locations of the ocean garbage patches. We further adapt these methods to decompose the surface ocean into almost-invariant sets in a forward-time and backward-time sense. We are also able to interpret the backward-time partitioning as a decomposition of the surface ocean into basins of attraction of the major garbage patches.

Our modelling framework allows for the possibility that particles of water may exit the ocean’s surface by washing up on coastlines or being absorbed into the polar sea ice. Dynamically speaking, we have an open dynamical system. Our approach also allows us to determine
the probability that a particle in any given location will eventually leave the computational domain.

An outline of this paper is as follows. In Section II we describe the data and a method to discretize the dynamics. We use this method to calculate the forward evolution of trajectories and surface up- and downwelling; these results are contained in Section III. In Section IV we define attracting sets, basins of attraction and absorption probabilities for the discretized dynamics, and explain how we will use these to analyze the connectivity of the ocean. In Section V we compare these results with a different method of analyzing connectivity, based on a transfer operator method of constructing almost-invariant sets and basins of attraction. Section VI concludes.

II. DATA DESCRIPTION, DEFINITIONS AND METHOD

Let \( \bar{X} \) denote the entire surface of the ocean, considered to be a compact two-dimensional manifold. Denote by \( T(x) \) the terminal point of a trajectory beginning at \( x \in \bar{X} \) and integrated forward over 48 weeks using the (time-dependent) horizontal velocity vector fields taken from the Ocean General Circulation Model for the Earth Simulator (OFES model). OFES is a global high-resolution ocean-only model\(^{13,14} \) configured on a 1/10\(^{\circ} \) horizontal resolution grid with 54 vertical levels and forced with observed winds from the NCEP/NCAR reanalysis. Figure 1 shows the mean speed of the surface flow (in red) and the mean direction of the surface flow (arrows). The means are computed as an average over time at fixed spatial points and thus represent Eulerian information as opposed to the semi-Lagrangian analysis carried out in the rest of this paper.

Our computational domain \( X \subset \bar{X} \) consists of a two-dimensional horizontal slice of the ocean at a depth of 5m over the region extending from 75\(^{\circ} \)S to 75\(^{\circ} \)N. Forward orbits of \( X \) may permanently leave \( X \) via beaching or being frozen into the Arctic or Antarctic, so \( X \subset \bar{T}(X) \). We form \( T := T|_X : X \circ \), the restriction of \( T \) to \( X \), and refer to \( (X, T) \) as an open dynamical system, in contrast to the closed dynamical system \( (\bar{X}, \bar{T}) \).

In order to study the ocean’s connectivity, we first form a spatial discretization of the dynamics, using a method known as Ulam’s method\(^{15} \); see also\(^{16,17} \). We first grid the space \( X \) into boxes \( \{B_i\}_{i=1}^N \). We use \( 2^6 \times 2^6 \) boxes; discarding boxes \( B_i \not\subset X \) leaves us with \( N = 10234 \) boxes. Let \( \mathcal{I} := \{1, \ldots, N\} \), write \( X_N := \{B_i : i \in \mathcal{I}\} \) and define the collection of all sets that are unions of boxes in \( X_N \) by \( \mathbf{B}_N \). The proportion of mass in \( B_i \) mapped to \( B_j \) under one application of \( T \) is equal to

\[
P_{ij} := \frac{\text{area}(B_i \cap T^{-1}(B_j))}{\text{area}(B_i)}, \quad i, j \in \mathcal{I},
\]

where area is normalized over \( X \). The transition matrix \( P \) defines a Markov chain representation \( P \) of the dynamics, with the entries \( P_{ij} \) equal to the conditional transition probabilities between boxes. In practice, the entries of \( P \) must be numerically approximated using ocean trajectory data. Within the OFES vector fields, virtual particles are advected with a 4th-order Runge-Kutta method, using the Connectivity Modeling System v1.1\(^{18} \). We initialize a set of particles on \( X \) at \( t = 1 \) Jan 2001, uniformly distributed over a 0.2\(^{\circ} \) x 0.2\(^{\circ} \) grid (approx. 100 particles per box; there are fewer points in boxes that contain some land mass). We numerically estimate the entries of \( P \) by calculating

\[
P_{ij} = \frac{\#\{x : x \in B_i \text{ and } T(x) \in B_j\}}{\#\{x \in B_i\}}.
\]

In order to maintain a reasonably even sampling of points we reinitialize the uniform distribution of particles every 8 weeks to create 6 collections of consecutive 8-week trajectories and 6 transition matrices \( P^{(1)}, P^{(2)}, \ldots, P^{(6)} \), as in\(^4 \). We then form \( P = P^{(1)} \cdot P^{(2)} \cdot \ldots \cdot P^{(6)} \).

III. FORWARD EVOLUTION OF TRAJECTORIES AND SURFACE UP- AND DOWNWELLING

We can use \( P \) to visualize the forward evolution of a uniformly distributed set of points over the ocean’s surface. Let \( a \) be a vector with entries \( a_i := \text{area}(B_i), i \in \mathcal{I} \), and calculate \( a^{(k)} := aP^k \) for \( k \in \{0, 1, 2, \ldots\} \). Figure 2 depicts \( a^{(2)}, a^{(10)}, a^{(100)}, a^{(1000)} \), and we can observe a divergence of mass from the Equator and a convergence toward the centres of the subtropical gyres. Comparable results were obtained using a similar method in\(^{1,4} \).

We can also calculate amounts of upwelling and downwelling over 48 weeks by imposing the restriction that the surface area of the ocean is preserved. Thus, if \( a_i^{(1)} > a_i \) then the difference \( a_i^{(1)} - a_i \) must have been pushed down below the ocean’s surface (downwelled). Similarly, if \( a_i > a_i^{(1)} \) then the difference \( a_i - a_i^{(1)} \) must have emerged at the surface (upwelled). Ekman theory\(^{19} \), linearly relates the strength of the wind stress to the mass flux in
the upper 10 to 100 meters of the ocean. In our study, we assume that the thickness of the Ekman layer was a constant 50 metres over the entire ocean. We thus think of the ocean surface area as a horizontal layer of 50 metres depth. To calculate upwelling in the standard units of metres per day, we compute:

$$\frac{50 \max\{a_i - a_{i}^{(1)}, 0\}}{(7 \times 48) a_i},$$

where the factor $7 \times 48$ accounts for the number of days in 48 weeks. To calculate downwelling, we use $a_{i}^{(1)} - a_i$ in place of $a_i - a_{i}^{(1)}$.

In Figure 3a we observe a large amount of upwelling occurring around the equator, the Western coastal regions of North and South America, and the Western coastal regions of Africa. Downwelling occurs in the North and South Pacific, the Indian, and the North and South Atlantic oceans (see Figure 3b), and is closely related to the regions where the garbage patches are located\(^4\). The numerical values of both upwelling and downwelling are consistent with recent studies; see Figure 2b\(^2\), for example.

IV. ATTRACTING SETS, BASINS OF ATTRACTION, AND ABSORPTION PROBABILITIES

In this section we define the objects that we will use to analyze the connectivity of the surface ocean: attracting sets, basins of attraction and absorption probabilities.

A. Dynamical systems and attracting sets

We will use a definition of attracting sets and basins of attraction based on Milnor\(^2\).

**Definition 1:** Let $A^c := X \setminus A$. A set $A$ for which $\text{area}(A \cap T^{-1}(A^c)) = 0$ is called forward invariant. A forward invariant set $A$ for which $\text{area}(A^c \cap T^{-1}(A)) > 0$ is called attracting. The basin of attraction $D_A \subset X$ for an attracting set $A$ is defined as the set of points whose forward orbits tend toward $A$.

$$D_A := \{x \in X : d(T^k(x), A) \to 0 \text{ as } k \to \infty\},$$

where $d(x, A) = \inf\{\text{dist}(x, y) : y \in A\}$.
B. Markov chains and absorption probabilities

In Section II we defined a Markov chain representation of the dynamics, with the conditional transition probabilities defined by (1). We will now use this representation to relate attracting sets for dynamical systems to absorbing states for Markov chains.

**Definition 2:** We refer to the set of states \( S_k \subset \mathcal{I} \) as an absorbing closed communicating class if one has:

(i) For each pair \( i,j \in S_k \), there exists an \( m \geq 0 \) such that \( P_{ij}^m > 0 \).

(ii) \( P_{ij} = 0 \) for all \( i \in S_k, j \notin S_k \), and

(iii) \( P_{ij} > 0 \) for at least one \( i \notin S_k, j \in S_k \).

Property (i) says that each class is communicating (there is a positive probability of moving between any pairs of states in the class in a finite number of steps); property (ii) says that each class is closed (there is a zero probability of moving out the class to another state elsewhere in our computational domain \( X \)); we do not, however, restrict movement from states in the class to states outside of \( X \), and allow the possibility that \( \sum_j P_{ij} < 1 \) for \( i \in S_k \); property (iii) says that each class is absorbing (there is a positive probability of external states moving into the class in \( m \) steps). The notation \( S_k \) allows for the possibility that there are \( K \) absorbing closed communicating classes, each identified by an index number \( k \in \{1, \ldots, K \} \). We can connect Definitions 1 and 2 according to the following simple Lemma:

**Lemma 1:** If \( S_k \) is an absorbing closed communicating class for the Markov chain defined by \( P \) in (1), then \( A_k := \bigcup_{i \in S_k} B_i \) is an attracting set for the open dynamical system \( T : X \to X \).

**Proof:** See Appendix A.

Markov chain theory also allows us to compute the probability, denoted \( h_{k,i} \), of eventually hitting an absorbing closed communicating class \( S_k \), conditioned on beginning in state \( i \notin S_k \). As each class \( S_k \), \( k = 1, \ldots, K \) is closed, clearly \( h_{k,i} = 0 \) for \( i \in \bigcup_{j=1}^K S_j \) so we are only interested in \( h_{k,i} \) for \( i \notin \bigcup_{j=1}^K S_j \).

The dynamical systems interpretation of \( h_{k,i} \) is the proportion of particles beginning in \( B_i \notin A_k \) that eventually hit the attracting set \( A_k := \bigcup_{j \in S_k} B_j \). A result adapted from 22 and used by 23 in a dynamical systems context, provides a method for exactly computing the vector of absorption probabilities \( h = (h_{k,i}) \). We extend the method of 23 in two ways: firstly, we allow for the possibility that particles in the attracting sets may leak out of the domain, and secondly, we handle attracting sets consisting of multiple states. We first define \( \mathcal{I}_S := \bigcup_k S_k \), \( \mathcal{I}_T := \mathcal{I} \setminus \mathcal{I}_S \), and \( \{\hat{P}_k\}_{k \in \{1, \ldots, K\}} \) by

\[
\hat{P}_k = \begin{bmatrix}
\Pi_k P_k & 0 \\
R_k & Q_k
\end{bmatrix},
\]

where \( P_{k,ij} = P_{ij} \) for \( i,j \in S_k \), \( \Pi_k \) is a diagonal matrix of size \( |S_k| \) with \( \Pi_{k,ii} = 1/\sum_{j \in S_k} P_{ij} \) for \( i \in S_k \), \( R_k,ij = P_{ij} \) for \( j \in S_k, i \in \mathcal{I}_T \), and \( Q_{k,ij} = P_{ij} \) for \( i,j \in \mathcal{I}_T \).

**Theorem 1:** The vector of absorption probabilities \( h_k = (h_{k,i})_{i \in \mathcal{I}_S \cup \mathcal{I}_T} \) into an absorbing closed communicating class \( S_k \) is the minimal nonnegative solution to

\[
\hat{P}_k g = g.
\]

where \( g \) is a vector constrained to have \( g_i = 1 \) for \( i \in S_k \).

**Proof:** See Appendix B.

We also define the vector \( H = (H_i), i = \mathcal{I}_T \) by

\[
H_i := \arg \max_k h_{k,i}.
\]

The value of \( H_i \) is the index \( k \) that corresponds to the absorbing communicating class \( S_k \) that a Markov chain beginning in state \( i \) is most likely to hit. Using the numerical approximation of \( P_{ij} \) given in (2), \( H_i \) allows us to identify the attracting set \( \bigcup_{j \in S_k} B_j \) that particles \( x \in B_i \)
are most likely to hit when evolved by the Markov dynamics. Thus $H$ is a natural way to identify ocean regions whose long-term behaviour is similar.

The following algorithm summarizes the steps necessary to define $P$, calculate the attracting sets and the probability of absorption into each of them:

**Algorithm 1:**

1. Partition the computational domain $X$ into connected sets $\{B_1, B_2, \ldots, B_N\}$.
2. Construct the transition matrix $P$ corresponding to the open system, following (2).
3. Determine the absorbing closed communicating classes of $P$. The communicating classes may be easily and quickly computed using e.g. Tarjan’s algorithm\textsuperscript{24}.
4. Compute the vectors $h_k$ by solving (6) (this is simple to implement in e.g. MATLAB using the back-slash command) and $H$ according to (7).

Applying Algorithm 1 to the global ocean’s surface in the OFES model, we identify 10 attracting regions: 5 in the North Pacific regions, 1 in the North Atlantic, 2 North of Alaska, 1 off the coast of Peru and 1 in the Southern Ocean. The probabilities of absorption into each of these attracting regions are depicted in Figure 4. The region $(106, 108) \times (-46, -42)$ off the coast of Peru (depicted in Figure 4b) is strongly attracting over a large portion of the surface ocean, especially over the South Pacific, the South Atlantic and Indian oceanic regions. The absorption probability into the region at $(-144, -142) \times (62, 64)$ (depicted in Figure 4g) is close to 0.1 over the North Atlantic, indicating that around 10% of the water in this region will eventually (over an infinite period of time) be absorbed into this attracting region. Particles in the region around the North Sea have some positive probability of being absorbed into the attracting region $(-74, -70) \times (72, 74)$ (depicted in Figure 4j); likewise, particles in the Southern Ocean have some positive probability of being absorbed into the attracting region $(158, 160) \times (-76, -74)$ (Figure 4a). With the exception of the attracting region off the coast of Peru, the remainder of the regions that we identify appear to be only very weakly attracting, with the probability of absorption being close to zero over much of the surface ocean.

Figure 5 shows the vector $H$ defined by (7). Particles beginning anywhere in the region shown in yellow (corresponding roughly to the North Atlantic) are more likely to be attracted to the attracting region in the North Atlantic (shown in Figure 4g) than any other attracting set; particles beginning in a region near the Arctic shown in dark red are more likely to be attracted to the attracting region in the Arctic (shown in Figure 4j) than any other attracting set, and particles in the remainder of the ocean, encompassing the North and South Pacific, the South Atlantic, the Indian and the Southern regions, are more likely to be attracted to the attracting region near Peru (shown in Figure 4b) than any other attracting set. From an oceanographic perspective, Figure 4 shows that most of the ocean outside the Arctic and North Atlantic is well-connected. Although the precise locations of the individual attracting regions might be different in different models or different years, the large-scale structure agrees with earlier results. The work of\textsuperscript{24} showed that there is very limited exchange of surface water between the North Atlantic Ocean and the rest of the oceans, so we would expect that to show up in vector $h$. The result that the region North of the Bering Strait is separate from the Pacific and Atlantic Ocean is likely because the OFES model does not include the currents in the Arctic north of 75N, and water can therefore not flow from the Bering Strait into the Atlantic.

**V. EIGENVECTOR METHODS**

In this section we present alternative approaches for analysing the global dynamics of the OFES model. Our strategy is to compute eigenvectors of the matrix $P$ corresponding to real eigenvalues of $P$ close to 1.

**A. Left eigenvectors of $P$**

To motivate our approach, let us first consider an idealised situation where the transition matrix $P$ generated from the OFES model contains $M$ single-box absorbing states and 5 garbage patches, each of which are absorbing closed communicating classes, and further that the basins of attraction are pairwise disjoint. In this idealised situation, we also assume that there is no loss of trajectories through beaching or being frozen in ice, so the matrix $P$ is row-stochastic. In such a situation, the leading eigenvalue of $P$ is 1, this eigenvalue has multiplicity $M + 5$, and one may find a basis of eigenvectors supported on the (disjoint) box collections comprising the basins of attraction. The matrix $P$ has a block diagonal structure consisting of $M$ diagonal unit-sized blocks with entries 1, and 5 further larger blocks corresponding to the 5 garbage patches.

Suppose now that we perturb $P$ continuously; by classical matrix perturbation theory (Theorem II.5.1\textsuperscript{25}), one knows that the $M + 5$ copies of the eigenvalue 1 will move continuously as a group, possibly becoming $M + 5$ distinct eigenvalues nearby 1. Further, under such a perturbation, the span of the $M + 5$-dimensional eigenspace corresponding to the eigenvalue 1 also moves continuously to a nearby group of eigenspaces of the perturbed $P$ of total dimension $M + 5$. Arguing in this way, we expect the (signed) supports of the top $M + 5$ eigenvectors of the perturbed $P$ to reveal to us (through linear combinations) the supports of the unperturbed $P$. Precursors of these ideas are discussed\textsuperscript{26} in the simpler situation of a row-stochastic, reversible $P$, and methods are put
FIG. 4: Absorption probabilities into the attracting regions. Locations of the attracting regions are marked with pink crosses, and the coordinates of the boxes containing the crosses ($2 \times 2$ regions) are listed underneath each figure.
South Atlantic Oceans in the North and South Pacific, Indian, and North and South Atlantic regions clearly highlight five ocean garbage patches, present change between larger collections of boxes. The eigenvalues close to 1, whose eigenvalues correspond to small exchange between large collections of boxes. The eigenvalues of \( \mathbf{P} \) are listed in Table I.

In practice, we visualise the eigenvectors to determine their supports. Figure 5 shows left eigenvectors of \( \mathbf{P} \) close to 1, whose eigenvalues correspond to small exchange between large collections of boxes. The eigenvectors clearly highlight five ocean garbage patches, present in the North and South Pacific, Indian, and North and South Atlantic Oceans, consistent with Figure 2d. Dymanically, this makes sense because there is likely to be small exchange between the (attracting) garbage patches.

In this particular case study the top four eigenvectors (not shown) highlighted combinations of single boxes related to the (possibly leaky) absorbing states. Further down the spectrum at positions 5, 6, 7, 9, the eigenvectors show the slow-exchange dynamics between the garbage patches (Figure 6). Some of these eigenvectors highlight the patches along with some of the individual absorbing states, for example Figure 6a shows a small highlighted region on the southwest coast of South America, in addition to two garbage patches in the South Pacific and South Atlantic.

### B. Right eigenvectors of \( \mathbf{P} \)

Let us now turn to the right eigenvectors of \( \mathbf{P} \). Clearly we have the same complex mix of eigenvalues as discussed above. Under left multiplication, the matrix \( \mathbf{P}^\top \) is the matrix representation of the dual dynamical action of \( \mathbf{P} \); see Lemma 5\(^{29}\). Thus, the right eigenvectors \( \mathbf{v} \) of \( \mathbf{P} \) associated with real eigenvalues \( \lambda \neq 1 \) close to 1 have similar properties to the left eigenvectors, with two important differences: they capture \textit{backward-time} dynamics and rather than spanning a space that looks approximately like the long-term mass distribution \( \mathbf{p} \) (in the OFES model, this distribution is concentrated in the single-box absorbing states and in the garbage patches) restricted to subregions, they span a space that looks approximately like \( \mathbf{I} \) restricted to subregions.

Moreover, by comparison with Theorem 1, one can interpret the indicated regions as \textit{basins of attraction}. Let us consider again the idealised situation of Section VA: the transition matrix \( \mathbf{P} \) has \( M \) single-box absorbing states and 5 garbage patches, each of which is an absorbing closed communicating class, with pairwise disjoint basins of attraction. Further, there is no loss of trajectories so that \( \mathbf{P} \) is stochastic. There will be \( M + 5 \)

\[
\begin{array}{l|ll}
\lambda & \mathbf{P} & \hat{\mathbf{P}} \\
\hline
\lambda_1 & 1.0000 & 1.0000 \\
\lambda_2 & 1.0000 & 1.0000 \\
\lambda_3 & 1.0000 & 1.0000 \\
\lambda_4 & 0.9999 & 1.0000 \\
\lambda_5 & 0.9999 & 0.9999 \\
\lambda_6 & 0.9996 & 0.9999 \\
\lambda_7 & 0.9991 & 0.9996 \\
\lambda_8 & 0.9975 & 0.9991 \\
\lambda_9 & 0.9913 & 0.9975 \\
\lambda_{10} & 0.9852 & 0.9913 \\
\lambda_{11} & 0.9838 & 0.9852 \\
\lambda_{12} & 0.9826 & 0.9838 \\
\lambda_{13} & 0.9680 & 0.9826 \\
\lambda_{14} & 0.9645 & 0.9680 \\
\end{array}
\]
FIG. 6: Selected left eigenvectors of $P$ showing the locations of the five great ocean garbage patches.

C. Left eigenvectors of $\hat{P}$: OFES model

Finally, we wish to create a forward-time partition of the ocean surface analogous to the backward-time partition shown in Figure 7. To do this, we require something
akin to a time-reversal of the transition matrix $P$. The standard way to do this is to use the invariant measure $\mu$ (see e.g.
31 and in the dynamical systems setting32). The problem with the OFES model is that there is not a single $\mu$, but at least 12 (conditional) invariant densities, corresponding to the distinct absorbing states, consisting of individual boxes.

Rather than consider the (multiple) infinite futures of surface particles, we use the 1000-year future, namely $\eta := a^{(1000)}$ the 1000-year pushforward of a uniform measure over the ocean (depicted in the final panel of Figure 2) as a proxy for $\mu$. The matrix $P_{ij} := \eta_j P_{ji} / \eta_i$ represents an approximate backward-time dynamics under left multiplication. The vector $\eta$ represents an approximate invariant measure of the dynamics of $P$, since $\sum_i \eta_i P_{ij} = \eta_j \sum_i P_{ji}$, and $\sum_i P_{ji}$ is approximately a vector of 1s (it is not exactly 1 because of the tracer leakage). We now compute the right eigenvectors of $\hat{P}_{ij}$, which contain information about the almost-invariant regions for the flow in forward-time, and importantly span a space that looks approximately like 1 restricted to subregions. The eigenvalues corresponding to $\hat{P}$ are given in Table I.

In Figure 8 we present the four right eigenvectors of $\hat{P}$ that correspond to the four eigenvectors shown in Figures 6 and 7. In Figure 8a, one can identify a yellow patch extending outwards from South America to the east, and a deep red triangular patch extending west, corresponding to the South Atlantic and South Pacific garbage patches. In Figure 8b, several coloured regions are shown, each corresponding to a distinct garbage patch. In Figures 8c and 8d one sees the deep red region in the North Atlantic and the deep red region extending out from the Indian Ocean to the east, past southern Australia and South America.

Figure 8 shows some interesting features of inter-ocean connections. The main structures visible Figure 8a are in the southeast Pacific, in an area roughly overlapping with the cold tongue there33 but also including a bit of the South Atlantic around Drake Passage. There is a second structure in Figure 8a in the South Atlantic, that interestingly, excludes the Benguela upwelling34, which according to Figure 8d is more connected to a large structure encompassing all of the Indian Ocean, the southwestern part of the Pacific and nearly all of the Southern Ocean. In the tropics, the Indonesian Throughflow (the phenomenon by which water from the Pacific flows into the Indian Ocean via the Indonesian Archipelago35) is also visible in both Figure 8b and 8c, as is the movement of Pacific waters into the South Atlantic through Drake Passage at the Southern tip of South America in these same panels.

VI. CONCLUSION

Ulam’s method, now a staple tool in many dynamical systems applications, enables one to use Markov chain ideas to analyse dynamical systems. Employing this method, we built a Markov representation of the dynamics of the global surface ocean. We were then able to compute the evolution of densities over many centuries,
calculate surface upwelling and downwelling, and identify attracting regions using strongly connected components, exploiting a connection between attracting sets for dynamical systems and absorbing closed communicating classes of Markov chains. We then calculated the probability of eventual absorption into an attracting region from anywhere on the surface ocean. We used the absorption probabilities as a basis for partitioning the ocean into different non-overlapping regions, by calculating which attracting set a given fluid element is most likely to be eventually absorbed into. Finally, we were able to interpret the left and right eigenvectors of the Markov chain transition matrix as almost-invariant sets and basins of attraction, respectively. We used these eigenvector techniques as a powerful method of identifying garbage patch locations and mapping their basins of attraction. We thus dynamically decomposed the global ocean surface into weakly interacting parts in both forward and backward time.

Decompositions such as these one can form the basis of a dynamical geography of the ocean surface, where the boundaries between basins are determined from the circulation itself, rather than from arbitrary geographical demarkations. One of the results that comes out of this decomposition, for instance, is that the Atlantic and Pacific Oceans are split into Northern and Southern parts at roughly the Equator, but that the Indian Ocean is one entity from 30°S to 30°N. Maps of these kinds of features in the ocean’s dynamical geography allow us to better understand how the different ocean basins are connected.

VII. ACKNOWLEDGEMENTS

GF is supported by an ARC Future Fellowship. RMS is supported by an Australian Postgraduate Award and the ARC Centre of Excellence for the Mathematics and Statistics of Complex Systems (MASCOS). EvS is supported by the Australian Research Council via grant DE130101336

Appendix A: Proof of Lemma 1

Only properties (ii) and (iii) of Definition 1 are required. Suppose $S_k$ is an absorbing closed communicating class for the Markov chain with conditional transition probabilities given by (1). Then $P_{ij} = 0$ for $i \in S_k$, $j \notin S_k$, which implies that $\text{area}(A \cap T^{-1}A_j) = 0$ for all $j \notin S_k$, so $A$ is a forward invariant set. Also, $P_{ij} > 0$ for $j \in S_k$ and at least one $i \notin S_k$, which implies that $\text{area}(A_i \cap T^{-1}(A)) > 0$ for at least one $i \notin S_k$, so $A$ is an attracting set according to Definition 1.

Appendix B: Proof of Theorem 1

The proof is an expanded version of the proof of Theorem 1.3.22; we present the proof here for completeness to demonstrate the application to the substochastic setting.
Let \((Z_t)_{t \in \mathbb{N}}\) be the substochastic Markov process over state space \(I\), with conditional transition probabilities
\[
P(Z_{t+1} = j | Z_t = i) = P_{ij}
\]
for \(i, j \in I\).

Firstly note that if \(i \in S_k\) then \(h_{k,i} = P\{Z_{t+0} \in S_k | Z_t = i\} = 1\). Next we examine the case \(i \in I_T\). We have
\[
h_{k,i} := P(X_{t+r} \in S_k, \text{ some } r \geq 1 | X_t = i) = \frac{P(X_{t+r} \in S_k, \text{ some } r \geq 2, X_{t+1} \in I_T, X_t = i)}{P(X_t = i)} = \sum_{j \in I_T} P(X_{t+r} \in S_k, \text{ some } r \geq 2, X_{t+1} = j, X_t = i) = \sum_{j \in I_T} \sum_{k \in S_k} P_{ij} \cdot P_{kj} = \sum_{j \in I_T} h_{k,j} \cdot P_{ij} + \sum_{j \in S_k} P_{ij}
\]

Therefore, for \(i \in I_T\), we have shown that \(h_{k,i}, i \in I_T\) is a solution to the problem:

\[
\sum_{j \in I_T \cup S_k} P_{ij} g_j = g_i, \text{ subject to } g_j = 1 \text{ for } j \in S_k.
\]

(B1)

Now we show that \(h_{k,i}, i \in I_T\) is the minimal solution to (B1). Suppose that \(f\) is another solution, then \(f_i = 1\) for \(i \in S_k\) and for \(i \in I_T\) we have

\[
f_i = \sum_{j \in I_T \cup S_k} P_{ij} f_j = \sum_{j \in S_k} P_{ij} + \sum_{j \in I_T} P_{ij} f_j
\]

(B2)

Substituting \(f_j = \sum_{\ell \in I_T \cup S_k} P_{j\ell} f_\ell\) in the final term,

\[
f_i = \sum_{j \in S_k} P_{ij} + \sum_{j \in I_T} P_{ij} \left( \sum_{\ell \in S_k} P_{j\ell} + \sum_{\ell \in I_T} P_{j\ell} f_\ell \right)
\]

\[
= \sum_{j \in I_T} P_{ij} + \sum_{j \in I_T} \sum_{\ell \in S_k} P_{ij} P_{j\ell} + \sum_{j \in I_T} \sum_{\ell \in I_T} P_{ij} P_{j\ell} f_\ell
\]

\[
= P\{Z_{t+1} \in S_k | Z_t = i\} + P\{Z_{t+2} \in S_k | Z_t = i, Z_{t+1} \in I_T\} + \sum_{j \in I_T} P_{ij} P_{j\ell} f_\ell
\]

(B3)

After repeating the substitution for \(f\) in the final term \(n\) times, we obtain

\[
f_i = P\{Z_{t+1} \in S_k | Z_t = i\} + \ldots + P\{Z_{t+n} \in S_k | Z_t = i, Z_{t+1}, \ldots, Z_{t+n-1} \in I_T\} + \sum_{j_1, \ldots, j_n \in I_T} P_{i,j_1} P_{j_1,j_2} \ldots P_{j_{n-1},j_n} f_{j_n}
\]

(B4)

If \(f \geq 0\) then the last term on the right is non-negative and the remaining terms sum to \(P\{Z_{t+r} \in S_k | 0 \leq r \leq n, Z_t = i\}\) (the probability of hitting \(S_k\) within \(n\) steps). So

\[
f_i = \lim_{n \to \infty} P\{Z_{t+r} \in S_k \text{ some } 0 \leq r \leq n | Z_t = i\}
\]

\[
= P\{Z_{t+r} \in S_k \text{ some } r \geq 0 | Z_t = i\} = h_{k,i}.
\]

REFERENCES

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